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A note on gradient Young measure relaxation of Dieudonné-Rashevsky type control problems with integrands $f(s, \xi, v)$

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1. Introduction and main result.

In calculus of variations or optimal control, relaxation of a given problem means to define a new problem, whose feasible domain contains the original one (possibly in the sense of imbedding), whose objective is lower semicontinuous with respect to a suitable topology, and whose minimal value is the same as in the original problem. [01] Consequently, the relaxed problem admits global minimizers and can be accessed by direct methods. [02] In optimal control, the proof of the necessary optimality conditions in the form of the Pontryagin principle is based on the relaxation of the given problem as well. [03]
There are two well-introduced approaches for the relaxation of variational or control problems. Assuming that the objective of the problem reads as

$$F(x) \;=\; \int_\Omega f(s, x(s), Jx(s))\, ds \tag{1.1}$$

with $\Omega \subset \mathbb{R}^m$ and $f(s,\xi,v)\colon\ \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R} \cup \{\,(+\infty)\,\}$, the first approach requires the replacement of the integrand f — depending on the dimensions m and n — by its convex or quasiconvex envelope with respect to the variable v [04] while the feasible domain of the original problem remains unchanged. The second approach is the introduction of generalized controls (Young measures) $\boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K}) \subset L^\infty\big[\,\Omega\,,\ rca\,(\mathbb{R}^{nm})\,\big]$ (see Section 2 below). [05] Here arises a new problem with the functional

$$\widetilde{F}(x, \boldsymbol{\mu}) \;=\; \int_\Omega \int_{\mathbb{R}^{nm}} f(s, x(s), v)\, d\mu_s(v)\, ds \tag{1.2}$$

and the additional constraint

$$\partial x_i(s)/\partial s_j \;=\; \int_{\mathbb{R}^{nm}} v_{ij}\, d\mu_s(v) \ \text{ for a. a. } s \in \Omega,\ 1 \leqslant i \leqslant n\,,\ 1 \leqslant j \leqslant m\,. \tag{1.3}$$

The aim of the present paper is to pursue the second approach and to provide a relaxation theorem for *multidimensional control problems of Dieudonné-Rashevsky type* in terms of Young measures. Recently, problems of this kind have found fruitful applications in mathematical image processing [06] but they arise as well in the geometric theory of convex bodies, [07] in material sciences [08] or in underdetermined boundary value

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[01] See e. g. [Buttazzo 89] , pp. 2 ff. and pp. 16 ff., and [Gamkrelidze 78] .

[02] Cf. [Dacorogna 08] , p. 3.

[03] We refer to [Ioffe/Tichomirow 79] , pp. 85 ff. and 213 ff., and [Ginsburg/Ioffe 96] , p. 92, Definition 3.2. and Theorem 3.3.

[04] Cf. [Dacorogna 08] , p. 416 ff., Theorem 9.1., and p. 432, Theorem 9.8. and Remark 9.9., (i).

[05] [Gamkrelidze 78] , pp. 21 ff. and 135 ff., and [Pedregal 97] , pp. 10 ff. and 133 ff.

[06] [Angelov 11] , [Brune/Maurer/Wagner 09] , [Franek/Franek/Maurer/Wagner 12] , [Wagner 09d] , [Wagner 10] and [Wagner 11a] .

[07] [Andrejewa/Klötzler 84a] , [Andrejewa/Klötzler 84b] , p. 149 f.

[08] [Lur'e 75] , pp. 240 ff., [Ting 69a] , p. 531 f., [Ting 69b] , [Wagner 96] , pp. 76 ff.

problems for implicit first-order PDE's. [09] In abstract form, they may be stated as follows:

$$(\mathrm{P})_1: \quad F(x,u) = \int_\Omega f(s, x(s), u(s))\, ds \longrightarrow \inf!\,; \quad (x,u) \in W_0^{1,p}(\Omega, \mathbb{R}^n) \times L^p(\Omega, \mathbb{R}^{nm})\,; \tag{1.4}$$

$$Jx(s) = \begin{pmatrix} \partial x_1(s)/\partial s_1 & \dots & \partial x_1(s)/\partial s_m \\ \vdots & & \vdots \\ \partial x_n(s)/\partial s_1 & \dots & \partial x_n(s)/\partial s_m \end{pmatrix} = u(s) \in \mathrm{K} \subset \mathbb{R}^{nm} \ \text{ for a. a. } s \in \Omega \tag{1.5}$$

where $n \geqslant 1$, $m \geqslant 1$, $\Omega \subset \mathbb{R}^m$ is a bounded (strongly) Lipschitz domain and $\mathrm{K} \subset \mathbb{R}^{nm}$ is a convex body containing the origin in its interior.
In analogy to the basic problem of multidimensional calculus of variations, the control problem $(\mathrm{P})_1$ requires quasiconvex instead of convex relaxation unless $n = 1$ or $m = 1$. In previous research, the author pointed out that the appropriate tool for the quasiconvex relaxation of $(\mathrm{P})_1$ is the *lower semicontinuous quasiconvex envelope* $f^{(qc)}(s, \xi, v)$ of the integrand f with respect to the variable v (cf. Definition 3.2. below) [10] but provided a Young measure relaxation theorem in a special case only. [11] The open question how the relaxation of $(\mathrm{P})_1$ with a general integrand $f(s, \xi, v)$ can be expressed in terms of Young measures will be answered in the following Theorem 1.1.

Theorem 1.1. (Young measure relaxation of $(\mathrm{P})_1$**)** *Assume that the data in* $(\mathrm{P})_1$ *satisfy the following properties:* $n \geqslant 1$, $m \geqslant 1$, $1 \leqslant p < \infty$, $\Omega \subset \mathbb{R}^m$ *is a bounded Lipschitz domain with* $\mathfrak{o} \in \mathrm{int}\,(\Omega)$, $\mathrm{K} \subset \mathbb{R}^{nm}$ *is a convex body with* $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$, *and the integrand* $f(s, \xi, v)\colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R} \cup \{\, (+\infty) \,\}$ *belongs to the class* $\widetilde{\mathfrak{F}}_{\mathrm{K}}$ *described in Definition 3.3. below. Together with* $(\mathrm{P})_1$, *we consider two further control problems:*

$$(\mathrm{P})_2: \quad F^{(qc)}(x,u) = \int_\Omega f^{(qc)}(s, x(s), u(s))\, ds \longrightarrow \inf!\,; \quad (x,u) \in W_0^{1,p}(\Omega, \mathbb{R}^n) \times L^p(\Omega, \mathbb{R}^{nm})\,; \tag{1.6}$$

$$G(x,u) = Jx(s) - u(s) = 0 \quad (\forall)\, s \in \Omega\,; \quad u(s) \in \mathrm{K} \quad (\forall)\, s \in \Omega\,; \tag{1.7}$$

$$(\mathrm{P})_3: \quad \widetilde{F}(x, \boldsymbol{\mu}) = \int_\Omega \int_{\mathrm{K}} f(s, x(s), v)\, d\mu_s(v)\, ds \longrightarrow \inf!\,; \quad (x, \boldsymbol{\mu}) \in W_0^{1,p}(\Omega, \mathbb{R}^n) \times \boldsymbol{\mathcal{G}}(\mathrm{K})\,; \tag{1.8}$$

$$\widetilde{G}(x, \boldsymbol{\mu}) = Jx(s) - \int_{\mathrm{K}} v\, d\mu_s(v) = 0 \quad (\forall)\, s \in \Omega \tag{1.9}$$

where $\boldsymbol{\mathcal{G}}(\mathrm{K}) \subset L^\infty\big[\,\Omega\,,\, rca\,(\mathrm{K})\,\big]$ *is the set of gradient Young measures described in Definition 2.1., 2) below. Denote by* m_1, m_2 *and* m_3 *the minimal values of the problems* $(\mathrm{P})_1$, $(\mathrm{P})_2$ *and* $(\mathrm{P})_3$, *respectively. Then*

1) The three minimal values m_1, m_2 *and* m_3 *coincide.*

2) The problems $(\mathrm{P})_2$ *and* $(\mathrm{P})_3$ *admit global minimizers. Moreover, if* $(\hat{x}, \hat{u})$ *is a global minimizer of* $(\mathrm{P})_1$ *or* $(\mathrm{P})_2$ *then* $(\,\hat{x}\,,\,\{\,\delta_{\hat{u}(s)}\,\}\,)$ *is a global minimizer of* $(\mathrm{P})_3$.

The main ingredient for the proof is a characterization theorem for gradient Young measures supported on K (Theorem 2.9.) arising as an appropriate generalization of a result of Kinderlehrer/Pedregal. In view of its importance, we will provide a complete proof of this theorem as well. Two further elements of the proof of Theorem 1.1. are Wagner's representation theorem for the lower semicontinuous quasiconvex envelope, which expresses $f^{(qc)}(s, \xi, v)$ as the minimum of the values $\langle\, f(s, \xi, v)\,,\, \nu\,\rangle$ where ν runs through a certain set of probability measures (Theorem 3.8.), and Schäl's measurability theorem for the "optimal" selector (Theorem 4.4.).

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09) [Dacorogna/Marcellini 97], [Dacorogna/Marcellini 98] and [Dacorogna/Marcellini 99].

10) [Wagner 09a] – [Wagner 09c] and particularly [Wagner 11b], pp. 192 ff.

11) [Wagner 09c], p. 617, Theorem 4.2., with $f = f(v)$.

The structure of the paper is as follows: After closing this introduction with a synopsis of notations, we will collect in Section 2 the needed facts about gradient Young measures and then prove the announced characterization theorem. In Section 3, we will summarize the properties of unbounded quasiconvex functions and the lower semicontinuous quasiconvex envelope of the integrand. Finally, we turn in Section 4 to the investigation of the Young measure relaxed control problem $(\mathrm{P})_3$ and to the proof of Theorem 1.1.

Notations.

We denote by $C^k(\Omega, \mathbb{R}^r)$, $L^p(\Omega, \mathbb{R}^r)$ and $W^{k,p}(\Omega, \mathbb{R}^r)$ the spaces of r-dimensional vector functions whose components are k-times continuously differentiable, belong to $L^p(\Omega, \mathbb{R})$ or to the Sobolev space of $L^p(\Omega, \mathbb{R})$-functions with weak derivatives up to kth order in $L^p(\Omega, \mathbb{R})$, respectively ($k \in \{0, 1, \ldots\}$, $1 \leqslant p \leqslant \infty$). Moreover, $W_0^{1,p}(\Omega, \mathbb{R}^r)$ denotes the subspace of the compactly supported functions within $W^{1,p}(\Omega, \mathbb{R}^r)$. The components of $x \in W_0^{1,\infty}(\Omega, \mathbb{R}^r)$ will be considered as Lipschitz functions with zero boundary values. [12] The symbols x_{s_j} and $\partial x / \partial s_j$ may denote the classical as well as the weak partial derivative of x by s_j. Jx denotes the Jacobi matrix of the function x. $rca\,(\mathrm{K})$ denotes the space of all (signed) Radon measures supported on K, and $rca^{\,pr}\,(\mathrm{K}) \subset rca\,(\mathrm{K})$ denotes the subset of probability measures on K. The notions for Young measures will be introduced in Subsection 2.a) below.
The extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{(+\infty)\}$ will be equipped with the natural topological and order structures where $(+\infty)$ is the greatest element. Throughout the paper, for all functions $f\colon\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$, which are allowed to take the value $(+\infty)$, we will assume that the effective domain $\mathrm{dom}\,(f) = \{ v \in \mathbb{R}^{nm} \mid f(v) < (+\infty) \}$ is nonempty. The restriction of the function f to the subset A of its range of definition is denoted by $f \,\big|\, \mathrm{A}$. A convex body $\mathrm{K} \subset \mathbb{R}^{nm}$ will be understood as a convex, compact set with nonempty interior. [13]
Finally, we will frequently use *three nonstandard notations.* "$\{x^N\}$, A" denotes a sequence $\{x^N\}$ with members $x^N \in \mathrm{A}$. If $\mathrm{A} \subseteq \mathbb{R}^r$ then the abbreviation "$(\forall)\, t \in \mathrm{A}$" has to be read as "for almost all $t \in \mathrm{A}$" resp. "for all $t \in \mathrm{A}$ except a r-dimensional Lebesgue null set". The symbol $\mathfrak{o}$ denotes, depending on the context, the zero element resp. the zero function of the underlying space.

2. Characterization of gradient Young measures on K.

a) Basic definitions and properties.

We consider the Bochner space $L^\infty\big[(\Omega), rca\,(\mathrm{K})\big]$ of weakly*-measurable, measure-valued maps $\boldsymbol{\mu}\colon\ \Omega \to rca\,(\mathrm{K})$, which is equipped with its weak*-topology. [14]

Definition 2.1. *1)* **(Young measures on K, generalized controls)** [15] *An element $\boldsymbol{\mu} \in L^\infty\big[(\Omega), rca\,(\mathrm{K})\big]$ is called a Young measure iff there exists a so-called generating sequence $\{u^N\}$, $L^\infty(\Omega, \mathbb{R}^{nm})$ with a) $u^N(s) \in \mathrm{K}\ (\forall)\, s \in \Omega\ \forall N \in \mathbb{N}$ and b) $\{\delta_{u^N(s)}\} \xrightarrow{\ *\ } \boldsymbol{\mu}$ within $L^\infty\big[(\Omega), rca\,(\mathrm{K})\big]$. Consequently, Young measures are precisely those elements of $L^\infty\big[(\Omega), rca\,(\mathrm{K})\big]$, which take values within the subset $rca^{\,pr}\,(\mathrm{K})$ of probability measures on K. The set of all Young measures on K is denoted by $\boldsymbol{\mathcal{Y}}(\mathrm{K})$.*

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12) [Evans/Gariepy 92], p. 131, Theorem 5.

13) See e. g. [Schneider 93].

14) We refer to [Edwards 65], pp. 557 ff. and 586 ff. In particular, by ibid., p. 590, Theorem 8.18.3., the duality relation $L^\infty\big[\Omega, rca\,(\mathrm{K})\big] \cong \big(L^1\big[(\Omega), C^0(\mathrm{K}, \mathbb{R})\big]\big)^*$ holds true. Consequently, on $L^\infty\big[(\Omega), rca\,(\mathrm{K})\big]$, the different measurability concepts for Bochner spaces agree, and the weak*-topology on the unit ball is metrizable.

15) See [Gamkrelidze 78] pp. 23 ff., and [Müller 99], pp. 115 ff.

2) **(Gradient Young measures on K, generalized gradient controls)** [16] *An element* $\boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$ *is called a gradient Young measure iff there exists a sequence* $\{ x^N \}$, $W^{1,\infty}(\Omega, \mathbb{R}^n)$ *with the properties a)* $Jx^N(s) \in \mathrm{K}$ $(\forall)\, s \in \Omega\ \forall N \in \mathbb{N}$ *and b)* $\{ \delta_{Jx^N(s)} \} \xrightarrow{\ *\ } \boldsymbol{\mu}$ *within* $L^\infty\big[\,(\Omega)\,,\, rca\,(\mathrm{K})\,\big]$. *The subset of all gradient measures is denoted by* $\boldsymbol{\mathcal{G}}(\mathrm{K}) \subset \boldsymbol{\mathcal{Y}}(\mathrm{K})$.

Definition 2.2. (First moment of a Young measure, underlying deformation) *For* $\boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$, *the measurable, essentially bounded function* $u\colon\ \Omega \to \mathbb{R}^{nm}$ *defined through*

$$u(s) \;=\; \int_{\mathrm{K}} v\, d\mu_s(v) \tag{2.1}$$

is called the first moment of $\boldsymbol{\mu}$.

Obviously, the first moment of a gradient Young measure is a gradient $u = Jx$ as well. The generating data $\{ x^N \}$, $W^{1,\infty}(\Omega, \mathbb{R}^n)$ of a gradient Young measure can be always chosen in such a way that a) $x^N \rightrightarrows \hat{x} \in W^{1,\infty}(\Omega, \mathbb{R}^n)$, b) $Jx^N \xrightarrow{\ *\ } J\hat{x}$ within $L^\infty(\Omega, \mathbb{R}^{nm})$ and $Jx^N(s)$, $J\hat{x}(s) \in \mathrm{K}$ $(\forall)\, s \in \Omega\ \forall N \in \mathbb{N}$, c) $\{ \delta_{Jx^N(s)} \} \xrightarrow{\ *\ } \boldsymbol{\mu}$ within $L^\infty\big[\,(\Omega)\,,\, rca\,(\mathrm{K})\,\big]$ and d) $J\hat{x}(s) = \int_{\mathrm{K}} v\, d\mu_s(v)$ $(\forall)\, s \in \Omega$.

Lemma 2.3. (Modification of generating sequences for gradient Young measures) *1) If* $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ *possesses a first moment* Jx *arising from a function* $x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ *then there exists a sequence* $\{ \widetilde{x}^N \}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ *with the properties of Definition 2.1., 2).*

2) Assume that there exist sequences $\{ w^N \}$, K *and* $\{ x^N \}$, $W^{1,\infty}(\Omega, \mathbb{R}^n)$ *with* $w^N \to w \in \mathrm{K}$, $w^N + Jx^N(s) \in \mathrm{K}$ $(\forall)\, s \in \Omega\ \forall N \in \mathbb{N}$ *and* $\{ \delta_{w^N + Jx^N(s)} \} \xrightarrow{\ *\ } \boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$. *If the first moment of* $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ *takes the shape* $w + Jx(s)$ *where* $x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ *then there exist sequences* $\{ \widetilde{w}^N \}$, $\mathrm{int}\,(\mathrm{K})$ *and* $\{ \widetilde{x}^N \}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ *with a)* $\widetilde{w}^N \to w$, *b)* $\widetilde{w}^N + J\widetilde{x}^N(s) \in \mathrm{K}$ $(\forall)\, s \in \Omega\ \forall N \in \mathbb{N}$ *and c)* $\{ \delta_{\widetilde{w}^N + J\widetilde{x}^N(s)} \} \xrightarrow{\ *\ } \boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$.

Proof. Part 1) is a special case of Part 2) with $w^N = w = \mathfrak{o}$. In order to prove 2), we may assume that the generating data for $\boldsymbol{\mu}$ fulfill $x^N \rightrightarrows x$ and $Jx^N \xrightarrow{\ *\ } Jx$. Using subdomains

$$\Omega^K = (1 - 1/K)\,\Omega \subset \Omega \tag{2.2}$$

and Lipschitz functions $\eta^K:\ \Omega \to \mathbb{R}$ with

$$\eta^K(s) \begin{cases} = 1 \ \mid\ s \in \Omega^K\,; \\ \in [\,0\,,\,1\,] \ \mid\ \text{else} \\ = 0 \ \mid\ s \in \partial\Omega\,; \end{cases} \qquad \text{and} \quad \big|\, \nabla\eta^K(s) \,\big| \,\leqslant\, C_1\, K \quad (\forall)\, s \in \Omega\,, \tag{2.3}$$

we define the functions $y^{N,K} \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ through

$$y^{N,K}(s) \;=\; x^N(s)\, \eta^K(s) \quad \Longrightarrow \quad Jy^{N,K}(s) \;=\; \eta^K(s)\, Jx^N(s) \;+\; x^N(s) \otimes \nabla\eta^K(s)\,. \tag{2.4}$$

From (2.3) it follows that

$$\big|\, x^N(s) \otimes \nabla\eta^K(s) \,\big| \;\leqslant\; C_1\, C_2\, K \cdot \sup_{s \,\in\, \Omega \,\setminus\, \Omega^K} \big|\, x^N(s) \,\big|\,, \tag{2.5}$$

and since x^N converges uniformly to $x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$, we may selcet a diagonal sequence $\{ y^{N(K),K} \}$ such that

$$\sup_{s \,\in\, \Omega \,\setminus\, \Omega^K} \big|\, x^{N(K)}(s) \,\big| \;\leqslant\; \frac{1}{C_1\, C_2\, K^2}\,. \tag{2.6}$$

[16] [Kinderlehrer/Pedregal 91], p. 333, and [Müller 99], p. 126, Definition 4.1.

Consequently, $w^{N(K)} + Jy^{N(K),K}(s) = w^{N(K)} + \eta^K(s)\, Jx^{N(K)}(s) + x^{N(K)}(s) \otimes \nabla\eta^K(s)$ belongs to $\mathrm{K} + B(\mathfrak{o}, 1/K)$ since $\eta^K(s) \leqslant 1$, and we find a monotonically increasing sequence of numbers $\lambda^K \in (\,0\,,\,1\,)$ with $\lim_{K\to\infty} \lambda^K = 1$ and $\lambda^K \left(w^{N(K)} + Jy^{N(K),K}(s) \right) \in \mathrm{K}\ (\forall)\, s \in \Omega\ \forall\, K \in \mathbb{N}$. Now the assertion of Part 2) is true with $\widetilde{w}^K = \lambda^K\, w^{N(K)} \in \mathrm{int}\,(\mathrm{K})$ and $\widetilde{x}^K = \lambda^K\, y^{N(K),K} \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$: We have $\widetilde{w}^K \to w$, $\widetilde{x}^K \rightrightarrows x$, $\widetilde{w}^K + J\widetilde{x}^K(s) \in \mathrm{K}\ (\forall)\, s \in \Omega\ \forall\, K \in \mathbb{N}$ and $J\widetilde{x}^K(s) - Jx^{N(K)}(s) \to 0\ (\forall)\, s \in \Omega$. It follows that $\{\, \delta_{\widetilde{w}^K + J\widetilde{x}^K(s)} \,\} \xrightharpoonup{\ *\ } \boldsymbol{\mu}$, and the proof is complete. ∎

The metrization of the weak*-topologies on $rca^{\,pr}\,(\mathrm{K})$ and $\boldsymbol{\mathcal{Y}}(\mathrm{K})$ can be described as follows:

Lemma 2.4. (Metrization of the weak*-topologies on $rca^{\,pr}\,(\mathrm{K})$ and $\boldsymbol{\mathcal{Y}}(\mathrm{K})$) *Assume that countably many functions $f_1 \equiv 1/|\,\Omega\,|$, $f_r \in C^0(\Omega, \mathbb{R}) \cap L^1(\Omega, \mathbb{R})$ with $\|\, f_r \,\|_{L^1(\Omega,\mathbb{R})} \cdot |\,\Omega\,| = 1$ for $r \geqslant 2$ as well as $g_l \in C^0(\mathrm{K}, \mathbb{R}) \cap W^{1,\infty}(\mathrm{K}, \mathbb{R})$ with $\|\, g_l \,\|_{C^0(\mathrm{K},\mathbb{R})} = 1$ and Lipschitz constants $L_l > 0$ for $l \geqslant 1$ are given such that $\{\, f_r \,\}$ resp. $\{\, g_l \,\}$ form dense subsets of the unit balls of $L^1(\Omega, \mathbb{R})$ resp. $C^0(\mathrm{K}, \mathbb{R})$ with respect to their norm topologies.*

1) [17] *Then the function $\sigma\colon\ rca^{\,pr}\,(\mathrm{K}) \times rca^{\,pr}\,(\mathrm{K}) \to \mathbb{R}$ defined by*

$$\sigma(\nu', \nu'') \;=\; \sum_{l=1}^{\infty} \frac{1}{2^l\,(1 + L_l)} \,\Big|\, \int_{\mathrm{K}} g_l(v)\,\big(\, d\nu'(v) - d\nu''(v) \,\big) \,\Big| \tag{2.7}$$

is a metrics on $rca^{\,pr}\,(\mathrm{K})$ with $\{\, \nu^N \,\}\,,\ rca^{\,pr}\,(\mathrm{K}) \xrightharpoonup{\ *\ } \nu \iff \sigma(\nu^N, \nu) \to 0$.

2) [18] *Further, the function $\varrho\colon\ \boldsymbol{\mathcal{Y}}(\mathrm{K}) \times \boldsymbol{\mathcal{Y}}(\mathrm{K}) \to \mathbb{R}$ defined by*

$$\varrho(\boldsymbol{\mu}', \boldsymbol{\mu}'') \;=\; \sum_{r=1}^{\infty} \sum_{l=1}^{\infty} \frac{1}{2^{r+l}\,(1 + L_l)} \,\Big|\, \int_{\Omega} \int_{\mathrm{K}} f_r(s)\, g_l(v)\,\big(\, d\mu'_s(v) - d\mu''_s(v) \,\big)\, ds \,\Big| \tag{2.8}$$

is a metrics on $\boldsymbol{\mathcal{Y}}(\mathrm{K})$ with $\{\, \boldsymbol{\mu}^N \,\}\,,\ \boldsymbol{\mathcal{Y}}(\mathrm{K}) \xrightharpoonup{\ *\ } \boldsymbol{\mu} \iff \varrho(\boldsymbol{\mu}^N, \boldsymbol{\mu}) \to 0$.

It follows that

$$\{\, \boldsymbol{\mu}^N \,\}\,,\ \boldsymbol{\mathcal{Y}}(\mathrm{K}) \xrightharpoonup{\ *\ } \boldsymbol{\mu} \iff \int_{\Omega} \int_{\mathrm{K}} f(s)\, g(v)\, d\mu_s^N(v)\, ds \to 0 \quad \forall\, f \in L^1(\Omega, \mathbb{R})\ \ \forall\, g \in C^0(\mathrm{K}, \mathbb{R})\,. \tag{2.9}$$

In particular, $\varrho(\boldsymbol{\mu}^N, \boldsymbol{\mu}) \to 0$ implies $\sigma(\mu_s^N, \mu_s) \to 0$ for almost all $s \in \Omega$. With respect to this topology, the sets $\boldsymbol{\mathcal{Y}}(\mathrm{K})$ and $\boldsymbol{\mathcal{G}}(\mathrm{K})$ are sequentially compact. [19] Obviously, the subsets $\big\{\, \{\, \delta_{u(s)} \,\} \in \boldsymbol{\mathcal{Y}}(\mathrm{K}) \;\big|\; u \in L^\infty(\Omega, \mathbb{R}^{nm})\,,\ u(s) \in \mathrm{K}\ \ (\forall)\, s \in \Omega \,\big\}$ and $\big\{\, \{\, \delta_{Jx(s)} \,\} \in \boldsymbol{\mathcal{G}}(\mathrm{K}) \;\big|\; x \in W^{1,\infty}(\Omega, \mathbb{R}^n)\,,\ Jx(s) \in \mathrm{K}\ \ (\forall)\, s \in \Omega \,\big\}$ lie dense in $\boldsymbol{\mathcal{Y}}(\mathrm{K})$ and $\boldsymbol{\mathcal{G}}(\mathrm{K})$, respectively.

b) The mean value theorem for gradient Young measures.

It is possible to assign to every gradient Young measure an "averaged" Young measure (with respect to s), which turns out to be a constant gradient Young measure. In the proof of the characterization theorem (Theorem 2.9.) below, the following proposition will be employed.

Proposition 2.5. (Mean value theorem in $\boldsymbol{\mathcal{G}}(\mathrm{K})$) [20] *Assume that $\Omega \subset \mathbb{R}^m$ is a bounded Lipschitz domain with $\mathfrak{o} \in \mathrm{int}\,(\Omega)$. We consider sequences $\{\, w^N \,\}\,,\ \mathrm{K}$ and $\{\, x^N \,\}\,,\ W_0^{1,\infty}(\Omega, \mathbb{R}^n)$, which satisfy a) $w^N \to$*

[17] [Wagner 09b], p. 446 f., Definition 2.1. and Theorem 2.2.

[18] [Wagner 09b], p. 448, Definition 2.4., and [Wagner 06], p. 52, Lemma 4.9.

[19] Cf. [Berliocchi/Lasry 73], p. 144, Proposition 1 (i), and [Wagner 09b], p. 450, Theorem 2.8., 2).

[20] [Wagner 09b], p. 450 f., Theorems 2.9. and 2.11.

$w \in \mathrm{K}$, b) $w^N + Jx^N(s) \in \mathrm{K}$ $(\forall)\, s \in \Omega$ $\forall\, N \in \mathbb{N}$, and c) $\{\, \delta_{w^N + Jx^N(s)} \,\} \xrightarrow{*} \boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$. Then there exists a further sequence $\{\, \widetilde{x}^N \,\}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ with the following properties:

1) $\lim_{N\to\infty} \|\, \widetilde{x}^N \,\|_{C^0(\Omega,\mathbb{R}^n)} \,=\, 0\,.$

2) $w^N + J\widetilde{x}^N(s) \in \mathrm{K}$ $(\forall)\, s \in \Omega$ $\forall\, N \in \mathbb{N}$.

3) $\{\, w^N + J\widetilde{x}^N \,\}$ generates a constant gradient Young measure $\boldsymbol{\nu} = \{\, \nu \,\} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$, which may be understood as the s-average of $\boldsymbol{\mu}$:

$$\lim_{N\to\infty} \int_\Omega g(\, w^N + Jx^N(s)\,)\, ds \,=\, \int_\Omega \int_{\mathrm{K}} g(v)\, d\mu_s(v)\, ds \tag{2.10}$$
$$= \lim_{N\to\infty} \int_\Omega g(\, w^N + J\widetilde{x}^N(s)\,)\, ds \,=\, \int_\Omega \int_{\mathrm{K}} g(v)\, d\nu(v)\, ds \quad \forall\, g \in C^0(\mathrm{K}, \mathbb{R})\,.$$

4) The first moment of $\{\, \nu \,\}$ is $w = \int_{\mathrm{K}} v\, d\nu$.

5) The average operator $A\colon\, \boldsymbol{\mathcal{G}}(\mathrm{K}) \to rca^{\,pr}(\mathrm{K})$ defined by $A(\boldsymbol{\mu}) = \nu$ is linear and continuous with respect to the weak-topologies: $\sigma(\, A(\boldsymbol{\mu}')\,,\, A(\boldsymbol{\mu}'')\,) \,\leqslant\, C\, \varrho(\boldsymbol{\mu}', \boldsymbol{\mu}'')$ $\forall\, \boldsymbol{\mu}',\, \boldsymbol{\mu}'' \in \boldsymbol{\mathcal{G}}(\mathrm{K})$.*

c) Characterization of gradient Young measures on K: disintegration and assembling.

In this subsection, we develop further ideas from [KINDERLEHRER/PEDREGAL 91] about disintegration and assembling of gradient Young measures. In their paper, they do not specified the range of the generating gradient sequences; consequently, we must check whether the constructions can be still performed under consideration of the additional gradient constraint $Jx(s) \in \mathrm{K}$ $(\forall)\, s \in \Omega$. This turns out to be possible for both the disintegration and the assembling theorem. While the the first theorem and its proof can be taken over without alterations, the proof of the latter requires some careful refinements.

Proposition 2.6. (Disintegration of gradient Young measures supported on K**)** [21] *Assume that $\mathfrak{o} \in \mathrm{int}\,(\Omega)$. If $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ then for almost all $s_0 \in \Omega$ there exists a sequence $\{\, y^N \,\}$, $W^{1,\infty}(\Omega, \mathbb{R}^n)$ with a) $Jy^N(s) \in \mathrm{K}$ $(\forall)\, s \in \Omega$ $\forall\, N \in \mathbb{N}$ and b) $\{\, \delta_{Jy^N(s)} \,\} \xrightarrow{*} \boldsymbol{\nu} \equiv \{\, \mu_{s_0} \,\}$ as a constant gradient Young measure. In short: A gradient Young measure takes almost everywhere values which occur in constant Young gradient measures.*

Proof. By definition, for $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ there exists a sequence $\{\, x^N \,\}$, $W^{1,\infty}(\Omega, \mathbb{R}^n)$ with $Jx^N(s) \in \mathrm{K}$ $\forall\, N \in \mathbb{N}$ $(\forall)\, s \in \Omega$ and $\{\, \delta_{Jx^N(s)} \,\} \xrightarrow{*} \boldsymbol{\mu}$. Fixing $s_0 \in \mathrm{int}\,(\Omega)$, we choose a number $K_0 \in \mathbb{N}$ such that $s_0 + s/K_0 \in \Omega$ $\forall\, s \in \Omega$. For all $K \geqslant K_0$, we consider the functions

$$y^{N,K}(s) \,=\, K \cdot \Big(x^N\big(\, s_0 + s/K\,\big) \,-\, x^N(s_0)\Big) \in W^{1,\infty}(\Omega, \mathbb{R}^n)\,, \tag{2.11}$$

which satisfy particularly

$$Jy^{N,K}(s) \,=\, Jx^N\big(\, s_0 + s/K\,\big) \in \mathrm{K} \quad \forall\, N \in \mathbb{N}\ \ \forall\, K \geqslant K_0\ \ (\forall)\, s \in \Omega\,. \tag{2.12}$$

Now it can be shown in complete analogy to [KINDERLEHRER/PEDREGAL 91] , p. 338 f., that an appropriate subsequence of $\{\, Jy^{N(K),K} \,\}$ generates the constant gradient Young measure $\boldsymbol{\nu} = \{\, \mu_{s_0} \,\}$ which, consequently, belongs to $\boldsymbol{\mathcal{G}}(\mathrm{K})$ together with $\boldsymbol{\mu}$. ∎

Proposition 2.7. (Assembling of gradient Young measures supported on K**)** [22] *Assume that $\mathfrak{o} \in \mathrm{int}\,(\Omega)$. Consider a Young measure $\boldsymbol{\mu} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$ with a) $\boldsymbol{\nu} \equiv \{\, \mu_s \,\}$ is a constant gradient Young measure for*

[21] Generalization of [KINDERLEHRER/PEDREGAL 91] , p. 338, Theorem 2.3.

[22] Generalization of [KINDERLEHRER/PEDREGAL 91] , p. 351, Theorem 6.1.

almost all $s \in \Omega$ *and b) there exists a function* $\hat{x} \in W^{1,\infty}(\Omega, \mathbb{R}^n)$ *with* $J\hat{x}(s) = \int_{\mathrm{K}} v \, d\mu_s(v) \in \mathrm{K}$ $(\forall)\, s \in \Omega$. *Then* $\boldsymbol{\mu}$ *belongs to* $\boldsymbol{\mathcal{G}}(\mathrm{K})$.

Proof. • Step 1. Lemma 2.8. [23] *For a bounded Lipschitz domain* $\Omega \subset \mathbb{R}^m$ *and a m-dimensional Lebesgue null set* $\mathrm{N} \subset \Omega$, *consider countably many functions* $f_{r,l} \in L^1(\Omega, \mathbb{R})$ *and* $r^K : \Omega \setminus \mathrm{N} \to (\,0\,,\,1\,)$. *Then there exist countably many points* $s^{K,M} \in \Omega \setminus \mathrm{N}$ *and numbers* $\varepsilon^{K,M} \in (\,0\,,\, r^K(s^{K,M})\,)$ *with the following properties:*

a) For all $K \in \mathbb{N}$, *the sets* $\big(\, s^{K,M} + \varepsilon^{K,M}\,\Omega \,\big) \subset \Omega$ *are mutually disjoint, and the set* $\mathrm{N}^K = \Omega \setminus \bigcup_{M=1}^{\infty} \big(\, s^{K,M} + \varepsilon^{K,M}\,\Omega \,\big)$ *has measure zero.*

b) For all $r,\, l \in \mathbb{N}$, *it holds that*
$$\int_\Omega f_{r,l}(s)\, ds \;=\; \lim_{K\to\infty} \sum_{M=1}^{\infty} f_{r,l}(s^{K,M}) \cdot \big|\, \varepsilon^{K,M}\,\Omega \,\big|\,. \tag{2.13}$$

• Step 2. *Choice of test functions.* We choose countable sets of test functions $\varphi_r \in L^1(\Omega, \mathbb{R}) \cap W^{1,\infty}(\Omega, \mathbb{R})$ and $g_l \in C^0(\mathrm{K}, \mathbb{R}) \cap W^{1,\infty}(\mathrm{K}, \mathbb{R})$ such that the family $\{\, \varphi_r \cdot g_l \,\}_{\,r,l}$ lies dense in $L^1\big[\,\Omega\,,\, C^0(\mathrm{K}, \mathbb{R})\,\big]$. Denote $\|\, \varphi_r \,\|_{C^0(\Omega,\mathbb{R})} = C_r$, $\|\, g_l \,\|_{C^0(\mathrm{K},\mathbb{R})} = C_l$ and the Lipschitz constants of φ_r and g_l by L_r and L_l. Using the "values" $\mu_s \in rca^{\,pr}\,(\mathrm{K})$ of $\boldsymbol{\mu}$ and the functions g_l, we define the functions
$$\widetilde{g}_l(s) \;=\; \int_{\mathrm{K}} g_l(v)\, d\mu_s(v)\,, \tag{2.14}$$
which, by measurability of $\boldsymbol{\mu}$, belong to $L^1(\Omega, \mathbb{R})$ for all $l \in \mathbb{N}$. Consequently, the functions
$$f_{r,l}(s) \;=\; \varphi_r(s) \cdot \widetilde{g}_l(s) \;=\; \int_{\mathrm{K}} \varphi_r(s)\, g_l(v)\, d\mu_s(v) \tag{2.15}$$
belong to $L^1(\Omega, \mathbb{R})$ as well.

• Step 3. *Consequences of the properties of the first moment* $J\hat{x}$ *of* $\boldsymbol{\mu}$. By Rademacher's theorem, [24] the Lipschitz function $\hat{x}$ is differentiable in all its components for all $s \in \mathrm{int}\,(\Omega) \setminus \mathrm{N}_0$ where the set $\mathrm{N}_0 \subset \Omega$ has measure zero. Then
$$\mathrm{N} \;=\; \partial\Omega \,\cup\, \mathrm{N}_0 \,\cup\, \Big\{\, s \in \mathrm{int}\,(\Omega) \;\Big|\; J\hat{x}(s) \neq \int_{\mathrm{K}} v\, d\mu_s(v) \;\text{ or }\; \{\, \mu_s \,\} \notin \boldsymbol{\mathcal{G}}(\mathrm{K}) \,\Big\} \tag{2.16}$$
is still a null set. Consequently, for every $s \in \Omega \setminus \mathrm{N}$ and every $K \in \mathbb{N}$, we find a number $0 < r^K(s) < 1/2^K$ with
$$\Big|\, \frac{\hat{x}(\,s + \varepsilon\, z\,) - \hat{x}(s)}{\varepsilon} - J\hat{x}(s)^{\mathrm{T}}\, z \,\Big| \;\leqslant\; \frac{1}{C_1\, K^2} \quad \forall\, z \in \Omega \quad \forall\, \varepsilon \in (\,0\,,\, r^K(s)\,) \quad \Longrightarrow \tag{2.17}$$
$$\big|\, \hat{x}(\,s + \varepsilon\, z\,) - \hat{x}(s) - \varepsilon\, J\hat{x}(s)^{\mathrm{T}}\, z \,\big| \;\leqslant\; \frac{\varepsilon}{C_1\, K^2} \quad \forall\, z \in \Omega \quad \forall\, \varepsilon \in (\,0\,,\, r^K(s)\,)\,. \tag{2.18}$$

Applying now Lemma 2.8. to the null set N from (2.16) and the families $\{\, f_{r,l} \,\}$ and $\{\, r^K \,\}$, we obtain families $\{\, s^{K,M} \,\}$, Ω and $\{\, \varepsilon^{K,M} \,\}$, $\mathbb{R}$ with $0 < \varepsilon^{K,M} < r^K(s^{K,M})$ and the properties a) and b) from above, which imply particularly that
$$\int_\Omega f_{r,l}(s)\, ds \;=\; \lim_{K\to\infty} \sum_{M=1}^{\infty} \varphi_r(s^{K,M})\, \widetilde{g}_l(s^{K,M}) \cdot \big|\, \varepsilon^{K,M}\,\Omega \,\big| \quad \forall\, r,\, l \in \mathbb{N}\,. \tag{2.19}$$

• Step 4. *A generating gradient sequence for* $\boldsymbol{\mu}$. With appropriate numbers $0 < c^K < 1$, we define subsets
$$\Omega^K \;=\; (1 - c^K)\,\Omega \subset \Omega \quad \text{with} \quad \big|\, \Omega^K \,\big| = \frac{K-1}{K}\, \big|\, \Omega \,\big| \tag{2.20}$$

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[23] [Kinderlehrer/Pedregal 91] , p. 351, Lemma 6.2.

[24] [Evans/Gariepy 92] , p. 81, Theorem 2.

and cutoff functions $\eta^K \in W^{1,\infty}(\Omega, \mathbb{R})$ with

$$\eta^K(s) \begin{cases} = 0 \;\mid\; s \in \Omega^K\,; \\ \in [\,0\,,\,1\,] \;\mid\; \text{else} \\ = 1 \;\mid\; s \in \partial\Omega\,; \end{cases} \qquad \text{and} \quad \big|\,\nabla\eta^K(s)\,\big| \leqslant 2\,K \quad (\forall)\, s \in \Omega\,. \tag{2.21}$$

By assumption, for the constant gradient Young measures $\boldsymbol{\nu}^{K,M} = \{\,\mu_{s^{K,M}}\,\}$, there are generating data $\{\,x^{N,K,M}\,\}$, $W^{1,\infty}(\Omega, \mathbb{R}^n)$ with $Jx^{N,K,M}(s) \in \mathrm{K}\ \forall\, N \in \mathbb{N}\ (\forall)\, s \in \Omega$, $x^{N,K,M} \rightrightarrows J\hat{x}(s^{K,M})^{\mathrm{T}}\, s$ and $\{\,\delta_{Jx^{N,K,M}(s)}\,\} \xrightarrow{\;*\;} \{\,\mu_{s^{K,M}}\,\}$. Define now functions (2.22)

$$y^K(s) \;=\; \begin{cases} \Big(\,\hat{x}(s^{K,M}) + \varepsilon^{K,M}\, x^{N(K,M),K,M}\Big(\,\dfrac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\Big) \cdot \Big(\,1 - \eta^K\Big(\,\dfrac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\Big) + \hat{x}(s) \cdot \eta^K\Big(\,\dfrac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big) \\ \hfill \mid\; s \in \big(\,s^{K,M} + \varepsilon^{K,M}\,\Omega\,\big) \\ \hat{x}(s) \;\mid\; \text{else} \end{cases}$$

wherein $N(K, M)$ will be specified in the estimates (2.26) and (2.41) below. Obviously, y^k belongs to $W^{1,\infty}(\Omega, \mathbb{R}^n)$ together with $\hat{x}$ and $x^{N(K,M),K,M}$ since η^K are Lipschitz as well. For the derivatives of y^K, we get

$$\begin{aligned} Jy^K(s) \;=\; & Jx^{N,K,M}\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\Big(\,1 - \eta^K\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\Big) + J\hat{x}(s) \cdot \eta^K\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big) \\ & + \frac{1}{\varepsilon^{K,M}} \cdot \Big(\,\hat{x}(s) - \hat{x}(s^{K,M}) - \varepsilon^{K,M}\, J\hat{x}(s^{K,M})^{\mathrm{T}}\,\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\Big) \otimes \nabla\eta^K\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big) \\ & + \Big(\,J\hat{x}\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big) - x^{N,K,M}\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\Big) \otimes \nabla\eta^K\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big) \end{aligned} \tag{2.23}$$

for almost all $s \in \big(\,s^{K,M} + \varepsilon^{K,M}\,\Omega\,\big)$ and $Jy^K(s) = J\hat{x}(s)$ else. The first two summands form a convex combination of elements of K, thus remaining within K. For the third summand, we infer from (2.18):

$$\begin{aligned} & \Big|\,\frac{1}{\varepsilon^{K,M}} \cdot \Big(\,\hat{x}(s) - \hat{x}(s^{K,M}) - \varepsilon^{K,M}\, J\hat{x}(s^{K,M})^{\mathrm{T}}\,\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\Big) \otimes \nabla\eta^K\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\,\Big| \\ & \leqslant \frac{1}{\varepsilon^{K,M}}\,\Big|\,\hat{x}(s) - \hat{x}(s^{K,M}) - \varepsilon^{K,M}\, J\hat{x}(s^{K,M})^{\mathrm{T}}\,\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\Big| \cdot C_1\,\Big|\,\nabla\eta^K\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\,\Big| && (2.24) \\ & \leqslant \frac{C_1}{\varepsilon^{K,M}}\,\big|\,\hat{x}(s^{K,M} + \varepsilon^{K,M}\,z) - \hat{x}(s^{K,M}) - \varepsilon^{K,M}\, J\hat{x}(s^{K,M})^{\mathrm{T}}\,z\,\big| \cdot 2\,K \;\leqslant\; \frac{2}{K} && (2.25) \end{aligned}$$

by substituting $z = (s - s^{K,M})/\varepsilon^{K,M}$. For the last summand, the uniform convergence of $x^{N,K,M}$ allows to find an index $N(K, M)$ such that

$$\begin{aligned} & \Big|\,\Big(\,J\hat{x}\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big) - x^{N,K,M}\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\Big) \otimes \nabla\eta^K\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\,\Big| \\ & \qquad \leqslant \Big|\,J\hat{x}\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big) - x^{N,K,M}\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\Big| \cdot C_1\,\Big|\,\nabla\eta^K\Big(\,\frac{s - s^{K,M}}{\varepsilon^{K,M}}\,\Big)\,\Big| \;\leqslant\; \frac{2}{K} \end{aligned} \tag{2.26}$$

for all $s \in \Omega$ and for all $N \geqslant N(K, M)$. Summing up, we find that

$$Jy^K(s) \;\in\; \mathrm{K} + \mathrm{B}(\mathfrak{o}, 4/K) \quad (\forall)\, s \in \Omega\,. \tag{2.27}$$

Consequently, there is a monotonically increasing sequence of numbers $\lambda^K \in (\,0\,,\,1\,)$ with $\lim_{K\to\infty} \lambda^K = 1$ such that $Jy^K \in \mathrm{K}$ can be guaranteed after replacing $x^{N(K,M),K,M}$ by $\lambda^K\, x^{N(K,M),K,M}$, thus shrinking the sum of the first two summands in (2.23).

• **Step 5.** *A suitable subsequence of* $\{ Jy^k \}$ *generates* $\boldsymbol{\mu}$. We investigate

$$I(r,l,K) \;=\; \int_\Omega \varphi_r(s)\, g_l(\, Jy^K(s)\,)\, ds \;=\; \sum_{M=1}^{\infty} \int_{(\, s^{K,M}+\varepsilon^{K,M}\,\Omega\,)} \varphi_r(s)\, g_l(\, Jy^K(s)\,)\, ds$$

$$= \sum_{M=1}^{\infty} \big(\, \varepsilon^{K,M}\,\big)^m \int_\Omega \varphi_r(\, s^{K,M} + \varepsilon^{K,M}\, z\,) \cdot g_l(\, Jy^K(\, s^{K,M} + \varepsilon^{K,M}\, z\,)\,)\, dz \tag{2.28}$$

$$= \sum_{M=1}^{\infty} \big(\, \varepsilon^{K,M}\,\big)^m \varphi_r(\, s^{K,M} + \varepsilon^{K,M}\, z^{K,M}\,) \int_\Omega g_l(\, Jy^K(\, s^{K,M} + \varepsilon^{K,M}\, z\,)\,)\, dz \tag{2.29}$$

with $z^{K,M} \in \Omega$, cf. [Wagner 06], p. 57. Inserting (2.23), we may continue as follows:

$$I(r,l,K) \;=\; \sum_{M=1}^{\infty} \big(\, \varepsilon^{K,M}\,\big)^m \varphi_r(\, s^{K,M} + \varepsilon^{K,M}\, z^{K,M}\,) \Big(\int_{\Omega^K} g_l(\, Jx^{N(K,M),K,M}(z)\,)\, dz \tag{2.30}$$

$$+ \int_{\Omega\setminus\Omega^K} g_l(\, Jy^K(\,...\,)\,)\, dz - \int_{\Omega\setminus\Omega^K} g_l(\, Jx^{N(K,M),K,M}(z)\,)\, dz \Big)$$

$$= T_1(r,l,K) + T_2(r,l,K) + T_3(r,l,K) \quad \text{with} \tag{2.31}$$

$$T_1(r,l,K) \;=\; \sum_{M=1}^{\infty} \big(\, \varepsilon^{K,M}\,\big)^m \varphi_r(s^{K,M}) \int_\Omega g_l(\, Jx^{N(K,M),K,M}(z)\,)\, dz\,; \tag{2.32}$$

$$T_2(r,l,K) \;=\; \sum_{M=1}^{\infty} \big(\, \varepsilon^{K,M}\,\big)^m \varphi_r(s^{K,M}) \Big(\int_{\Omega\setminus\Omega^K} g_l(\, Jy^K(\,...\,)\,)\, dz - \int_{\Omega\setminus\Omega^K} g_l(\, Jx^{N(K,M),K,M}(z)\,)\, dz \Big)\,; \tag{2.33}$$

$$T_3(r,l,K) \;=\; \sum_{M=1}^{\infty} \big(\, \varepsilon^{K,M}\,\big)^m \Big(\varphi_r(\, s^{K,M} + \varepsilon^{K,M}\, z^{K,M}\,) - \varphi_r(s^{K,M}) \Big) \cdot \Big(\,...\, \Big)\,. \tag{2.34}$$

Let us estimate first $T_2(r,l,K)$ and $T_3(r,l,K)$. For $T_2(r,l,K)$, we obtain from (2.20)

$$\big|\, T_2(r,l,K)\,\big| \;=\; \Big|\, \sum_{M=1}^{\infty} \big(\, \varepsilon^{K,M}\,\big)^m \varphi_r(s^{K,M}) \Big(\int_{\Omega\setminus\Omega^K} g_l(\, Jy^K(\,...\,)\,)\, dz - \int_{\Omega\setminus\Omega^K} g_l(\, Jx^{N(K,M),K,M}(z)\,)\, dz \Big) \Big| \tag{2.35}$$

$$\leqslant \Big(\sum_{M=1}^{\infty} \big(\, \varepsilon^{K,M}\,\big)^m \Big) \cdot C_r \cdot \big|\, \Omega \setminus \Omega^K \,\big| \cdot 2\, C_l \;\leqslant C_r\, C_l\, \frac{2}{K} \;=\; C_2(r,l) \cdot \frac{1}{K} \tag{2.36}$$

since Lemma 2.8. implies particularly that $\sum_{M=1}^{\infty} \big(\, \varepsilon^{K,M}\,\big)^m = 1$. Further, we get

$$\big|\, T_3(r,l,K)\,\big| \;\leqslant\; \Big|\, \sum_{M=1}^{\infty} \big(\, \varepsilon^{K,M}\,\big)^m \Big(\varphi_r(\, s^{K,M} + \varepsilon^{K,M}\, z^{K,M}\,) - \varphi_r(s^{K,M}) \Big) \tag{2.37}$$

$$\cdot \Big(\int_{\Omega^K} g_l(\, Jx^{N(K,M),K,M}(z)\,)\, dz \;+\; \int_{\Omega\setminus\Omega^K} g_l(\, Jy^K(\,...\,)\,)\, dz \Big) \Big|$$

$$\leqslant \Big(\sum_{M=1}^{\infty} \big(\, \varepsilon^{K,M}\,\big)^m \Big) \cdot L_r \cdot \varepsilon^{K,M} \cdot \sup_{z\in\Omega} \big|\, z\,\big| \cdot C_l\, \big|\, \Omega\,\big| \;\leqslant\; C_3(r,l)\, \varepsilon^{K,M} \;\leqslant\; C_3(r,l)\, \frac{1}{2^K}\,. \tag{2.38}$$

Further, for every $K \in \mathbb{N}$ we may determine an index $M(K)$ such that

$$\sum_{M=M(K)}^{\infty} \big(\, \varepsilon^{K,M}\,\big)^m \;\leqslant\; \frac{1}{K}\,. \tag{2.39}$$

For the finitely many indices $M = 1,\, ...\, ,\, M(K) - 1$, we modify the numbers $N(K,M)$ as follows: Denoting by $\mu^{N,K,M} \in rca^{\,pr}\,(\mathrm{K})$ the average of the Young measure $\{\, \delta_{Jx^{N,K,M}(s)}\,\}$ according to Proposition 2.5., the continuity of the average operator implies

$$\varrho\big(\, \{\, \delta_{Jx^{N,K,M}(s)}\,\}\, ,\, \{\, \mu_{s^{K,M}}\,\}\,\big) \to 0 \quad\Longrightarrow\quad \sigma\big(\, \mu^{N,K,M}\, ,\, \mu_{s^{K,M}}\,\big) \to 0 \tag{2.40}$$

where $\sigma(\,\cdot\,,\,\cdot\,)$ is defined as in Lemma 2.4. Now we may enlarge the numbers $N(K,M)$ until

$$\sigma\big(\,\mu^{N,K,M}\,,\,\mu_{s^{K,M}}\,\big)\;\leqslant\;\frac{1}{K}\quad\;\forall\,N\geqslant N(K,M) \tag{2.41}$$

holds. Consequently, we get

$$\Big|\int_{\mathrm{K}} g_l(v)\,d\mu^{N(K,M),K,M}(v)-\int_{\mathrm{K}} g_l(v)\,d\mu_{s^{K,M}}(v)\Big|\;\leqslant\;2^{l+1}\,(1+L_l)\cdot\frac{1}{K}\,. \tag{2.42}$$

Summing up, we obtain

$$\begin{aligned}
I(r,l,K)\;=\;&\sum_{M=1}^{\infty}\big(\,\varepsilon^{K,M}\,\big)^m\,\varphi_r(s^{K,M})\int_{\Omega}\int_{\mathrm{K}} g_l(v)\,d\mu_{s^{K,M}}(v)\,dz\;+\;T_2(r,l,K)\;+\;T_3(r,l,K) &\text{(2.43)}\\
&+\sum_{M=1}^{M(K)-1}\big(\,\varepsilon^{K,M}\,\big)^m\,\varphi_r(s^{K,M})\int_{\Omega}\Big(\int_{\mathrm{K}} g_l(v)\,d\mu^{N(K,M),K,M}(v)-\int_{\mathrm{K}} g_l(v)\,d\mu_{s^{K,M}}(v)\Big)\,dz\\
&+\sum_{M=M(K)}^{\infty}\big(\,\varepsilon^{K,M}\,\big)^m\,\varphi_r(s^{K,M})\int_{\Omega}\Big(\int_{\mathrm{K}} g_l(v)\,d\mu^{N(K,M),K,M}(v)-\int_{\mathrm{K}} g_l(v)\,d\mu_{s^{K,M}}(v)\Big)\,dz\quad\Longrightarrow
\end{aligned}$$

$$\begin{aligned}
&\Big|\,I(r,l,K)-\sum_{M=1}^{\infty}\big(\,\varepsilon^{K,M}\,\big)^m\,\varphi_r(s^{K,M})\int_{\Omega}\int_{\mathrm{K}} g_l(v)\,d\mu_{s^{K,M}}(v)\,dz\,\Big| &\text{(2.44)}\\
&\quad\leqslant\;\big|\,T_2(r,l,K)\,\big|\;+\;\big|\,T_3(r,l,K)\,\big|\;+\;\Big(\sum_{M=1}^{M(K)-1}\big(\,\varepsilon^{K,M}\,\big)^m\Big)\,C_r\cdot\big|\,\Omega\,\big|\cdot\Big|\int_{\mathrm{K}} g_l(v)\,d\mu^{N(K,M),K,M}(v)\\
&\qquad-\int_{\mathrm{K}} g_l(v)\,d\mu_{s^{K,M}}(v)\Big|\;+\;\Big(\sum_{M=M(K)}^{\infty}\big(\,\varepsilon^{K,M}\,\big)^m\Big)\,C_r\cdot\big|\,\Omega\,\big|\cdot 2\,C_l\\
&\quad\leqslant\;C_2(r,l)\,\frac{1}{K}+C_3(r,l)\,\frac{1}{K}+C_r\,\big|\,\Omega\,\big|\cdot 2^{l+1}\,(1+L_l)\frac{1}{K}+2\,C_r\,C_l\,\big|\,\Omega\,\big|\,\frac{1}{K}\quad\Longrightarrow &\text{(2.45)}\\
&\quad\lim_{K\to\infty}\Big|\,I(r,l,K)-\sum_{M=1}^{\infty}\big(\,\varepsilon^{K,M}\,\big)^m\,\varphi_r(s^{K,M})\int_{\Omega}\int_{\mathrm{K}} g_l(v)\,d\mu_{s^{K,M}}(v)\,dz\,\Big|\;=\;0\,. &\text{(2.46)}
\end{aligned}$$

Finally, Lemma 2.8., b) implies that

$$\begin{aligned}
&\lim_{K\to\infty}\sum_{M=1}^{\infty}\big(\,\varepsilon^{K,M}\,\big)^m\,\varphi_r(s^{K,M})\cdot\big|\,\Omega\,\big|\cdot\int_{\mathrm{K}} g_l(v)\,d\mu_{s^{K,M}}(v) &\text{(2.47)}\\
&=\;\lim_{K\to\infty}\sum_{M=1}^{\infty}\big(\,\varepsilon^{K,M}\,\big)^m\,\varphi_r(s^{K,M})\cdot\big|\,\Omega\,\big|\cdot\widetilde{g}_l(s^{K,M}) &\text{(2.48)}\\
&=\;\lim_{K\to\infty}\sum_{M=1}^{\infty}\varphi_r(s^{K,M})\,\widetilde{g}_l(s^{K,M})\,\big|\,\varepsilon^{K,M}\,\Omega\,\big|\;=\;\int_{\Omega}\int_{\mathrm{K}}\varphi_r(s)\,g_l(v)\,d\mu_s(v)\,ds\,. &\text{(2.49)}
\end{aligned}$$

Thus $\{\,Jy^K\,\}$ generates indeed $\boldsymbol{\mu}$, and the proof is complete. ∎

Propositions 2.6. and 2.7. can be combined into the following theorem:

Theorem 2.9. (Characterization of gradient Young measures on K**)** *A Young measure* $\boldsymbol{\mu}\in\boldsymbol{\mathcal{Y}}(\mathrm{K})$ *belongs to* $\boldsymbol{\mathcal{G}}(\mathrm{K})$ *iff its first moment is the gradient* Jx *of a function* $x\in W^{1,\infty}(\Omega,\mathbb{R}^n)$ *and almost all of its values* μ_s *arise from constant gradient Young measures belonging to* $\boldsymbol{\mathcal{G}}(\mathrm{K})$.

3. The lower semicontinuous quasiconvex envelope of the integrand $f(s,\xi,v)$.

a) Quasiconvex functions with unbounded values.

In the treatment of multidimensional control problems of Dieudonné-Rashevsky type, it is useful to work with an extended notion of quasiconvexity, which can be applied to functions taking the value $(+\infty)$ outside of the control domain $\mathrm{K}\subset\mathbb{R}^{nm}$. Consequently, we start with the following definition:

Definition 3.1. (Quasiconvex functions with unbounded values) [25] *A function* $f\colon\ \mathbb{R}^{nm}\to\overline{\mathbb{R}}$ *with the following properties is said to be quasiconvex:*

a) $\mathrm{dom}\,(f)\subseteq\mathbb{R}^{nm}$ *is a nonempty Borel set;*

b) $f\,\big|\,\mathrm{dom}\,(f)$ *is Borel measurable and bounded from below on every bounded subset of* $\mathrm{dom}\,(f)$*;*

c) for all $v\in\mathbb{R}^{nm}$*,* f *satisfies Morrey's integral inequality*

$$f(v)\ \leqslant\ \frac{1}{|\,\Omega\,|}\int_\Omega f(\,v+Jx(t)\,)\,dt\quad\forall\,x\in W_0^{1,\infty}(\Omega,\mathbb{R}^n)\,,\tag{3.1}$$

or equivalently

$$f(v)\ =\ \inf\Big\{\ \frac{1}{|\,\Omega\,|}\int_\Omega f(\,v+Jx(t)\,)\,dt\ \Big|\ x\in W_0^{1,\infty}(\Omega,\mathbb{R}^n)\,,\ v+Jx(t)\in\mathbb{R}^{nm}\ \ (\forall)\,t\in\Omega\,\Big\}\,.\tag{3.2}$$

Here $\Omega\subset\mathbb{R}^m$ *is a bounded Lipschitz domain.*

We adopt the convention that the integral $\int_{\mathrm{A}}(+\infty)\,dt$ takes the values zero or $(+\infty)$ if either $\mathrm{A}\subseteq\mathbb{R}^m$ is a m-dimensional Lebesgue null set or has positive measure. If $\mathrm{dom}\,(f)=\mathrm{K}$ is a convex body then the set of "test functions" within Morrey's integral inequality (3.2) allows for the obvious restriction [26]

$$f(v)\ =\ \inf\Big\{\ \frac{1}{|\,\Omega\,|}\int_\Omega f(\,v+Jx(t)\,)\,dt\ \Big|\ x\in W_0^{1,\infty}(\Omega,\mathbb{R}^n)\,,\ v+Jx(t)\in\mathrm{K}\ \ (\forall)\,t\in\Omega\,\Big\}\tag{3.3}$$

for all $v\in\mathrm{K}$. In the same spirit, the definition of the quasiconvex envelope can be generalized:

Definition 3.2. (Lower semicontinuous quasiconvex envelope $f^{(qc)}$ for functions with unbounded values) [27] *To a function* $f\colon\ \mathbb{R}^{nm}\to\overline{\mathbb{R}}$ *bounded from below, we define the lower semicontinuous quasiconvex envelope* $f^{(qc)}\colon\ \mathbb{R}^{nm}\to\overline{\mathbb{R}}$ *through* (3.4)

$$f^{(qc)}(v)\ =\ \sup\big\{\ g(v)\ \big|\ g\colon\ \mathbb{R}^{nm}\to\overline{\mathbb{R}}\ \ \textit{quasiconvex and lower semicontinuous,}\ g(w)\leqslant f(w)\ \ \forall\,w\in\mathbb{R}^{nm}\ \ \big\}\,.$$

This definition is motivated by the observation that any finite, quasiconvex function $g\colon\ \mathbb{R}^{nm}\to\mathbb{R}$ is continuous from the outset. [28] If a measurable function f is bounded from below and takes only values in $\mathbb{R}$ then Definition 3.2. coincides with the usual definition of the quasiconvex envelope, [29] and the function $f^{(qc)}$ is quasiconvex and continuous as well. In general, however, it is a matter of proof to ensure that $f^{(qc)}$ is a

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[25] [Wagner 09a] , p. 73, Definition 2.9., as a specification of [Ball/Murat 84] , p. 228, Definition 2.1., in the case $p=(+\infty)$. If f takes only values in $\mathbb{R}$ then Definition 3.1. agrees with the usual definition of quasiconvexity, cf. [Dacorogna 08] , p. 156 f., Definition 5.1., (ii).

[26] [Wagner 09a] , p. 74, Theorem 2.11., 2).

[27] [Wagner 09a] , p. 76, Definition 2.14., (2).

[28] [Dacorogna 08] , p. 159, Theorem 5.3., (iv).

[29] Cf. [Dacorogna 08] , p. 156 f., Definition 5.1., ii).

quasiconvex function in the sense of Definition 3.1. and is, consequently, admissible in the process of its own forming. If so, then $f^{(qc)}$ is the largest quasiconvex, lower semicontinuous function below f, and it satisfies the inequality $f^c(v) \leqslant f^{(qc)}(v) \leqslant f(v)$ for all $v \in \mathbb{R}^{nm}$. [30]

b) The lower semicontinuous quasiconvex envelope for $f(s, \xi, v)$.

In order to ensure the desired behaviour of the lower semicontinuous quasiconvex envelope, we specify the following function class $\widetilde{\mathcal{F}}_{\mathrm{K}}$.

Definition 3.3. (Function class $\widetilde{\mathcal{F}}_{\mathrm{K}}$) [31] *Let $\Omega \subset \mathbb{R}^m$ be a bounded Lipschitz domain and $\mathrm{K} \subset \mathbb{R}^{nm}$ a convex body with $\mathfrak{o} \in \operatorname{int}(\mathrm{K})$. We say that a function $f(s, \xi, v) \colon \Omega \times \mathbb{R}^n \times \mathbb{R}^{nm} \to \mathbb{R} \cup \{(+\infty)\}$ belongs to the class $\widetilde{\mathcal{F}}_{\mathrm{K}}$ iff there exists a m-dimensional Lebesgue null set $\mathrm{N} \subset \Omega$ with:*

a) $f(s, \xi, v) = (+\infty)$ for all $(s, \xi, v) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^n \times \big(\mathbb{R}^{nm} \setminus \mathrm{K}\big)$,

b) $f(s, \xi, v) < (+\infty)$ for all $(s, \xi, v) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^n \times \mathrm{K}$,

c) the restriction $f \,\big|\, \big(\big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^n \times \mathrm{K}\big)$ is Borel measurable with respect to s and continuous with respect to (ξ, v), and

d) f satisfies the growth condition

$$\big| f(s, \xi, v) \big| \leqslant A(s) + B(\xi, v) \quad \forall (s, \xi, v) \in \Omega \times \mathbb{R}^n \times \mathrm{K} \tag{3.5}$$

where $A \in L^1(\Omega, \mathbb{R})$, $A \,\big|\, \operatorname{int}(\Omega)$ is continuous, and B is bounded on every bounded subset of $\mathbb{R}^n \times \mathrm{K}$.

As a consequence, the lower semicontinuous quasiconvex envelope of $f \in \widetilde{\mathcal{F}}_{\mathrm{K}}$, which is formed with respect to the variable v, obeys the following properties:

Proposition 3.4. (Properties of $f^{(qc)}$ for $f \in \widetilde{\mathcal{F}}_{\mathrm{K}}$) [32] *Let $f \in \widetilde{\mathcal{F}}_{\mathrm{K}}$ be given. Then for every fixed $(s_0, \xi_0) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^n$ it holds that*

1) $f^c(s_0, \xi_0, v) \leqslant f^{(qc)}(s_0, \xi_0, v) \leqslant f(s_0, \xi_0, v)$ for all $v \in \mathbb{R}^{nm}$, which implies particularly $f^{(qc)}(s_0, \xi_0, v) = (+\infty)$ for all $v \in \mathbb{R}^{nm} \setminus \mathrm{K}$.

2) $f^{(qc)}(s_0, \xi_0, v) \colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is quasiconvex in the sense of Definition 3.1. Moreover, it is the largest lower semicontinuous, quasiconvex function below $f(s_0, \xi_0, v)$.

c) Representation of the lower semicontinuous quasiconvex envelope by probability measures.

It is well-known that the convex envelope of a function $f(s, \xi, v) \in \widetilde{\mathcal{F}}_{\mathrm{K}}$ with respect to the variable v admits the representation $f^c(s_0, \xi_0, w) = \operatorname{Min} \{ \int_{\mathrm{K}} f(s_0, \xi_0, v) \, d\nu(v) \,\big|\, \nu \in \mathrm{S}^c(w) \subset rca^{\,pr}(\mathrm{K}) \}$ where $\mathrm{S}^c(w) = \{ \nu \in rca^{\,pr}(\mathrm{K}) \,\big|\, \int_{\mathrm{K}} v \, d\nu(v) = w \}$. [00] For the lower semicontinuous quasiconvex envelope, this assertion has an analogon where ν is allowed to run through an appropriate subset $\mathrm{S}^{(qc)}(w) \subseteq \mathrm{S}^c(w)$ only. The following definitions and theorems establish a close connection between the lower semicontinuous quasiconvex envelope and gradient Young measures.

Definition 3.5. (The set-valued map $\mathrm{S}^{(qc)}$) [33] *For given $w \in \mathrm{K}$, the set $\mathrm{S}^{(qc)}(w) \subseteq rca^{\,pr}(\mathrm{K})$ consists of all probability measures $\nu \in rca(\mathrm{K})$ with the following properties: There exist sequences $\{ w^N \}$, $\operatorname{int}(\mathrm{K})$ and*

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[30] [Wagner 09a], p. 77, Theorems 2.18., (1) and 2.19.

[31] [Wagner 11b], p. 191, Definition 1.1., 2).

[32] [Wagner 11b], p. 198, Theorem 2.10., 1) and 2).

[00] Cf. [Wagner 09b], p. 444.

[33] Synopsis of [Wagner 09b], p. 452, Definition 3.1. and Lemma 3.2., and p. 459, Theorem 3.9., 2).

$\{x^N\}$, $W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ *with a)* $\lim_{N\to\infty} w^N = w$, *b)* $x^N \rightrightarrows \mathfrak{o}$, *c)* $w^N + Jx^N(s) \in \mathrm{K}$ $(\forall)\, s \in \Omega$ $\forall\, N \in \mathbb{N}$ *and d)* $\{\delta_{w^N + Jx^N(s)}\} \xrightarrow{\ *\ } \boldsymbol{\nu} \equiv \{\nu\} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ *as a constant gradient Young measure.*

Proposition 3.6. (Properties of $\mathrm{S}^{(qc)}(\,\cdot\,)$**)** *Let* $\mathrm{K} \subset \mathbb{R}^{nm}$ *be a convex body with* $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$*.*

1)[34] *The sets* $\mathrm{S}^{(qc)}(w)$ *are nonempty, weak*-closed and convex for all* $w \in \mathrm{K}$*.*

2)[35] *The set-valued map* $\mathrm{S}^{(qc)}\colon\ \mathrm{K} \to rca^{\,pr}\,(\mathrm{K})$ *is continuous on* $\mathrm{int}\,(\mathrm{K})$ *and upper semicontinuous on* K*.*

Proposition 2.7. and Theorem 2.9. may be complemented by the following assertion:

Proposition 3.7. (Characterization of constant gradient Young measures) $\boldsymbol{\nu} = \{\nu\} \in \boldsymbol{\mathcal{Y}}(\mathrm{K})$ *is a constant gradient Young measure iff there exists* $w \in \mathrm{K}$ *such that* $\nu \in \mathrm{S}^{(qc)}(w)$*.*

Proof. If $\boldsymbol{\nu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ is a constant gradient Young measure then its first moment takes the form

$$\int_{\mathrm{K}} v\, d\nu_s(v) \;\equiv\; w \in \mathrm{K} \tag{3.6}$$

by convexity of the integral.[36] Consequently, we may apply Lemma 2.3., 2) to the generating data of $\boldsymbol{\nu}$, thus confirming that ν matches Definition 3.5. with $\nu \in \mathrm{S}^{(qc)}(w)$. Conversely, any measure $\nu \in \mathrm{S}^{(qc)}(w)$ allows for the generation of a gradient Young measure $\{\nu\} = \boldsymbol{\nu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$. ∎

Now we may state the announced representation theorem for $f^{(qc)}(s,\xi,v)$.

Theorem 3.8. (Representation theorem for $f^{(qc)}$ **if** $f \in \widetilde{\mathfrak{F}}_{\mathrm{K}}$**)** *Assume that* $\Omega \subset \mathbb{R}^m$ *is a bounded Lipschitz domain,* $\mathrm{K} \subset \mathbb{R}^{nm}$ *is a convex body with* $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$*, and* $f(s,\xi,v)$ *is a function belonging to the class* $\widetilde{\mathfrak{F}}_{\mathrm{K}}$*. Then for every fixed* $(s_0,\xi_0) \in \big(\Omega \setminus \mathrm{N}\big) \times \mathbb{R}^n$ *and for all* $w \in \mathrm{K}$*, we have the representation*

$$f^{(qc)}(s_0,\xi_0,w) \;=\; \mathrm{Min}\,\Big\{ \int_{\mathrm{K}} f(s_0,\xi_0,v)\, d\nu(v) \;\Big|\; \nu \in \mathrm{S}^{(qc)}(w) \Big\}\,. \tag{3.7}$$

Proof. As a consequence of Definition 3.3., $f(s_0,\xi_0,v)$ is continuous on K as a function of v and $(+\infty)$ outside of K for almost all $s_0 \in \Omega$ and all $\xi_0 \in \mathbb{R}^n$. Now the assertion of Theorem 3.8. is implied by [Wagner 09b], p. 444, Theorem 1.4. ∎

4. Investigation of the relaxed control problem $(\mathrm{P})_3$.

a) The control-to-state mapping.

We now turn to the investigation of the relaxed control problem $(\mathrm{P})_3$ from Section 1. Our first observation is the continuity of the control-to-state mapping, which assigns to every feasible generalized control $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ that function $x \in W_0^{1,\infty}\,(\Omega,\mathbb{R}^n)$, which satisfies (1.9) together with $\boldsymbol{\mu}$.

Proposition 4.1. (Assignment of the feasible state as a linear, continuous operator) *We study* $(\mathrm{P})_3$ *in the analytical situation specified in Theorem 1.1. Then the operator* $T\colon\ \boldsymbol{\mathcal{G}}(\mathrm{K}) \to W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ *assigning to every feasible generalized control* $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ *the integral* $x \in W_0^{1,\infty}(\Omega,\mathbb{R}^n)$ *of its first moment* $Jx \in L^\infty(\Omega,\mathbb{R}^{nm})$ *is well-defined, linear and continuous.*

Proof. Consider first the operator $M\colon\ \boldsymbol{\mathcal{G}}(\mathrm{K}) \to L^\infty(\Omega,\mathbb{R}^{nm})$ assigning to $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ its first moment $u \in L^\infty(\Omega,\mathbb{R}^{nm})$. Obviously, M is linear. In order to prove continuity, we may assume without loss of

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34) [Wagner 09b] p. 452, Theorem 3.4., and p. 459, Theorem 3.10., 1).

35) [Wagner 09b], p. 452, Theorem 3.6., and p. 460, Theorem 3.12., 1).

36) [Bourbaki 52], Chap. IV, § 6, p. 204, Corollaire.

generality that the Lipschitz functions $g_{ij}(v) = v_{ij}$ are part of the sequence $\{ g_l \}$ used in the definition of the metrics $\varrho(\,\cdot\,,\,\cdot\,)$ and $\sigma(\,\cdot\,,\,\cdot\,)$ in Lemma 2.4. Then, applying the mean value theorem (Proposition 2.5.) to $\boldsymbol{\mu}'$, $\boldsymbol{\mu}'' \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ and denoting the average operator by A, we calculate

$$\big|\, u'(s) - u''(s) \,\big| \;=\; \Big|\int_{\mathrm{K}} v\,\big[\, d\mu'_s(v) - d\mu''_s(v) \,\big]\,\Big| \;\leqslant\; C\cdot\sum_{i,j}\Big|\int_{\mathrm{K}} v_{ij}\,\big[\, dA(\boldsymbol{\mu}')(v) - dA(\boldsymbol{\mu}'')(v) \,\big]\,\Big| \tag{4.1}$$

$$\leqslant\; \sigma\big(\, A(\boldsymbol{\mu}')\,,\, A(\boldsymbol{\mu}'') \,\big) \;\leqslant\; C\,\varrho(\,\boldsymbol{\mu}',\,\boldsymbol{\mu}''\,) \quad\Longrightarrow \tag{4.2}$$

$$\|\, M(\boldsymbol{\mu}') - M(\boldsymbol{\mu}'') \,\|_{L^\infty(\Omega,\mathbb{R}^{nm})} = \|\, u' - u'' \,\|_{L^\infty(\Omega,\mathbb{R}^{nm})} = \operatorname*{ess\,sup}_{s\,\in\,\Omega}\big|\, u'(s) - u''(s) \,\big| \leqslant C\,\varrho(\,\boldsymbol{\mu}',\,\boldsymbol{\mu}''\,)\,, \tag{4.3}$$

and M is continuous. If $\boldsymbol{\mu} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ is feasible in $(\mathrm{P})_3$ then its first moment takes the form $u = Jx$ with $x \in W^{1,\infty}_0(\Omega,\mathbb{R}^n)$, and the operator $I\colon\, L^\infty(\Omega,\mathbb{R}^{nm}) \to W^{1,\infty}_0(\Omega,\mathbb{R}^n)$ with $I(Jx) = x$ is well-defined and linear as well while its continuity follows from the Poincaré inequality. [37] Consequently, the composition $T = I\circ M$ is linear and continuous as well, and the proof is complete. ∎

b) Compactness of the feasible domain.

As a consequence of Proposition 4.1., we may confirm the compactness of the feasible domain $\mathcal{B}_3$ of $(\mathrm{P})_3$.

Proposition 4.2. (Boundedness of the feasible domain) *Consider again* $(\mathrm{P})_3$ *in the analytical situation specified in Theorem 1.1. Then the feasible domain* $\mathcal{B}_3 \subset W^{1,\infty}_0(\Omega,\mathbb{R}^n)\times\boldsymbol{\mathcal{G}}(\mathrm{K})$ *is bounded in the product of the* $W^{1,\infty}_0$*-norm topology and the metric topology on* $\boldsymbol{\mathcal{G}}(\mathrm{K})$ *generated by* ϱ*.*

Proof. Note first that, due to the assumptions about the functions f_r and g_l in Lemma 2.4., $\boldsymbol{\mathcal{Y}}(\mathrm{K})$ itself is bounded in the metrics (2.8) since

$$\varrho(\boldsymbol{\mu}',\boldsymbol{\mu}'') \;\leqslant\; \sum_{r=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{2^{r+l}\,(1+L_l)}\int_\Omega\big|\,f_r(s)\,\big|\int_{\mathrm{K}}\big|\,g_l(v)\,\big|\,\big(\,d\mu'_s(v) + d\mu''_s(v)\,\big)\,ds \tag{4.4}$$

$$\leqslant\; \sum_{r=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{2^{r+l}\,(1+L_l)}\int_\Omega 2\,\big|\,f_r(s)\,\big|\cdot\|\,g_l\,\|_{C^0(\mathrm{K})}\,ds \;\leqslant\; 2\sum_{r=1}^{\infty}\sum_{l=1}^{\infty}\frac{1}{2^{r+l}\,(1+L_l)} \;\leqslant\; 2\,. \tag{4.5}$$

Consequently, from (4.3), (4.5) and the Poincaré inequality we get for arbitrary $(x',\boldsymbol{\mu}')$, $(x'',\boldsymbol{\mu}'') \in \mathcal{B}_3$

$$\|\, x' - x'' \,\|_{W^{1,\infty}_0(\Omega,\mathbb{R}^n)} \;\leqslant\; C\,\|\, Jx' - Jx'' \,\|_{W^{1,\infty}_0(\Omega,\mathbb{R}^n)} \;\leqslant\; \widetilde{C}\,\varrho(\,\boldsymbol{\mu}',\,\boldsymbol{\mu}''\,) \;\leqslant\; 2\,\widetilde{C}\,, \tag{4.6}$$

and $\mathcal{B}_3$ is bounded together with $\boldsymbol{\mathcal{G}}(\mathrm{K}) \subset \boldsymbol{\mathcal{Y}}(\mathrm{K})$. ∎

Corollary 4.3. (Sequential compactness of the feasible domain) *Under the assumptions of Proposition 4.2., the feasible domain* $\mathcal{B}_3$ *of* $(\mathrm{P})_3$ *is sequentially compact in the product of the weak*-topologies on* $W^{1,\infty}_0(\Omega,\mathbb{R}^n)$ *and* $\boldsymbol{\mathcal{G}}(\mathrm{K})$*.*

Proof. From Proposition 4.2. and the sequential compactness of $\boldsymbol{\mathcal{G}}(\mathrm{K})$, we obtain immediately the sequential compactness of $\mathcal{B}_3$ in the weak*-product topology. ∎

c) Proof of Theorem 1.1.

For the convenience of the reader, we repeat SCHÄL's theorem about the measurability of the optimal selector.

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[37] [EVANS 98], p. 275, Theorem 1, holds true together with the Rellich-Kondrachov theorem even on a bounded Lipschitz domain, cf. [ADAMS/FOURNIER 07], p. 168, Theorem 6.3.

Theorem 4.4. (Measurability of the optimal selector) [38] *Let* $\Omega \subset \mathbb{R}^m$ *be the closure of a bounded domain. Assume that* $[\,\mathrm{X}\,,\,\sigma\,]$ *is a compact, separable metric space containing a countable, dense subset* $\widetilde{\mathrm{X}}$. *Recall that a measurable set-valued map* $\mathrm{S}\colon\,\Omega \to \mathfrak{P}(\mathrm{X})$ *with nonempty, closed images, whose intersections* $\mathrm{S}(s) \cap \widetilde{\mathrm{X}}$ *are dense in* $\mathrm{S}(s)$ *for all* $s \in \Omega$, *is called separable. Consider a Carathéodory function* $g(s,\nu)\colon\,\Omega \times \mathrm{X} \to \mathbb{R}$ *and a set-valued map* $\mathrm{S}\colon\,\Omega \to \mathfrak{P}(\mathrm{X})$ *with nonempty, closed images, which satisfy the following assumptions:*

a) for every $s \in \Omega$ *there exists an "optimal" element* $\hat{\nu} \in \mathrm{S}(s)$ *with* $g(s,\hat{\nu}) = \inf\{\, g(s,\nu) \mid \nu \in \mathrm{S}(s)\,\}$, *and*

b) the set-valued map S *admits an approximation by a sequence of separable set-valued maps* $\mathrm{S}^N\colon\,\Omega \to \mathfrak{P}(\mathrm{X})$ *with* $\lim_{N\to\infty} \mathrm{S}^N(s) = \mathrm{S}(s)$ *for all* $s \in \Omega$ *where the limit is taken in the sense of a Painlevé-Kuratowski.*[39]

Then there exists a Lebesgue measurable function $h\colon\,\Omega \to \mathrm{X}$ *(an "optimal selector") with*

$$h(s) \in \mathrm{S}(s) \quad \textit{and} \quad g(s,h(s)) \,=\, \inf\big\{\, g(s,\nu) \;\big|\; \nu \in \mathrm{S}(s)\,\big\}\,. \tag{4.7}$$

for all $s \in \Omega$.

We are now in position to prove the relaxation theorem (Theorem 1.1.) about the problems $(\mathrm{P})_1$, $(\mathrm{P})_2$ and $(\mathrm{P})_3$.

Proof of Theorem 1.1. • Step 1. $(\mathrm{P})_2$ *and* $(\mathrm{P}_3$ *admit finite minimal values and global minimizers.* For $(\mathrm{P})_2$, this has been already proven in [WAGNER 11B], p. 193, Theorem 1.4. For $(\mathrm{P})_3$, the finiteness of the minimal value is a consequence of Proposition 4.2. and the boundedness of the objective $\widetilde{F}\colon\, W_0^{1,\infty}(\Omega,\mathbb{R}^n) \times \boldsymbol{\mathcal{G}}(\mathrm{K}) \to \mathbb{R}$. Consequently, $(\mathrm{P})_3$ admits a minimizing sequence $\{\,(x^N,\boldsymbol{\mu}^N)\,\}$, and from Corollary 4.3. we conclude that we may pass to a feasible limit element $(\hat{x},\hat{\boldsymbol{\mu}})$ along a suitable subsequence $\{\,(x^{N'},\boldsymbol{\mu}^{N'})\,\}$. Moreover, by the Sobolev imbedding theorem,[40] we may assume that $x^{N'}$ converges uniformly, and thus $\lim_{N'\to\infty} \widetilde{F}(x^{N'},\boldsymbol{\mu}^{N'}) = \widetilde{F}(\hat{x},\hat{\boldsymbol{\mu}}) = m_3$ holds.

• Step 2. *The inequality* $m_2 \geqslant m_3$. Consider a minimizing sequence $\{\,(x^N,u^N)\,\}$ of $(\mathrm{P})_2$ with

$$m_2 + 1/N \,\geqslant\, F^{(qc)}(x^N,u^N) \,\geqslant\, m_2 \quad \forall\, N \in \mathbb{N}\,. \tag{4.8}$$

From Theorem 3.8., we get

$$F^{(qc)}(x^N,u^N) \,=\, \int_\Omega f^{(qc)}(s,x^N(s),u^N(s)\,)\,ds \,=\, \int_\Omega \underset{\nu\,\in\,\mathrm{S}^{(qc)}\left(\,u^N(s)\,\right)}{\mathrm{Min}} \int_{\mathrm{K}} f(s,x^N(s),v)\,d\nu(v)\,ds \tag{4.9}$$

$$= \int_\Omega \int_{\mathrm{K}} f(s,x^N(s),v)\,d\nu_s^N(v)\,ds \tag{4.10}$$

where $\nu_s^N \in rca^{\,pr}\,(\mathrm{K})$ is determined by

$$\int_{\mathrm{K}} f(s,x^N(s),v)\,d\nu_s^N(v) \,=\, \underset{\nu\,\in\,\mathrm{S}^{(qc)}\left(\,u^N(s)\,\right)}{\mathrm{Min}} \int_{\mathrm{K}} f(s,x^N(s),v)\,d\nu(v) \,=\, \underset{\nu\,\in\,\mathrm{S}^{(qc)}\left(\,Jx^N(s)\,\right)}{\mathrm{Min}} \int_{\mathrm{K}} f(s,x^N(s),v)\,d\nu(v)\,. \tag{4.11}$$

If the family $\boldsymbol{\nu}^N = \{\,\nu_s^N\,\}$ selected in (4.11) is measurable then, by Theorem 2.9., $\boldsymbol{\nu}^N$ belongs to $\boldsymbol{\mathcal{G}}(\mathrm{K})$ since every measure $\nu \in \mathrm{S}^{(qc)}(s)$ arises from a constant gradient Young measure $\{\,\nu\,\} \in \boldsymbol{\mathcal{G}}(\mathrm{K})$, and the first

[38] [SCHÄL 74], p. 220, Theorem 3.

[39] Cf. [AUBIN/FRANKOWSKA 90], p. 41, Definition 1.4.6.

[40] [ADAMS/FOURNIER 07], p. 85 f., Theorem 4.12., II.

moment of $\{\nu_s^N\}$ is given by $\int_{\mathrm{K}} v\,d\nu_s^N(v) = Jx^N(s)$ for almost all $s \in \Omega$. Then it would follow that the pair $(x^N, \{\nu_s^N\})$ is feasible in $(\mathrm{P})_3$, and we get

$$m_2 + 1/N \geqslant F^{(qc)}(x^N, u^N) = \widetilde{F}(x^N, \{\nu_s^N\}) \geqslant m_3 \tag{4.12}$$

for arbitrary $N \in \mathbb{N}$. Consequently, it must be checked whether it is possible to select a measurable family in (4.11). In order to confirm this, we apply Theorem 4.4. to the following data: $\Omega \subset \mathbb{R}^m$, the set $\mathrm{X} = rca^{\,pr}(\mathrm{K})$ which becomes a compact, separable metric space together with the metrics σ arising from the weak*-topology, the Carathéodory function $g(s,\nu) = \int_{\mathrm{K}} f(s, x^N(s), v)\,d\nu(v)$ and the set-valued map $\mathrm{S}^{(qc)}(u^N(\cdot))\colon \Omega \to \mathfrak{P}(rca^{\,pr}(\mathrm{K}))$. In fact, as proven in [Wagner 09c], p. 618, the latter satisfies assumptions a) and b) of Theorem 4.4. Thus it is possible to find a measurable family $h = \boldsymbol{\nu}^N = \{\nu_s^N\}$ with the property (4.11).

• **Step 3.** *The inequality* $m_3 \geqslant m_1 \geqslant m_2$. Consider now a minimizing sequence $\{(x^N, \boldsymbol{\mu}^N)\}$ for $(\mathrm{P})_3$ with

$$m_3 + 1/N \geqslant \widetilde{F}(x^N, \boldsymbol{\mu}^N) \geqslant m_3\,. \tag{4.13}$$

By Definition 2.1., 2) and Lemma 2.3., $\boldsymbol{\mu}^N \in \boldsymbol{\mathcal{G}}(\mathrm{K})$ can be approximated by gradient Young measures of the form $\{\delta_{Jx^{N,K}(s)}\}$ with $x^{N,K} \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$, and the continuity of the control-to-state mapping in $(\mathrm{P})_3$ (Proposition 4.1.) and the continuity of $\widetilde{F}$ with respect to both variables imply

$$m_3 + 1/N + 1/K \geqslant \widetilde{F}(x^{N,K}, \{\delta_{Jx^{N,K}(s)}\}) \geqslant m_3\,. \tag{4.14}$$

Consequently, we get

$$\begin{aligned} m_3 + 1/N + 1/K \geqslant \widetilde{F}(x^{N,K}, \{\delta_{Jx^{N,K}(s)}\}) &= \int_\Omega \int_{\mathrm{K}} f(s, x^{N,K}(s), v)\,d\delta_{Jx^{N,K}(s)}(v)\,ds \\ &= \int_\Omega f(s, x^{N,K}(s), Jx^{N,K}(s))\,ds = F(x^{N,K}, Jx^{N,K}) \geqslant m_1 \end{aligned} \tag{4.15}$$

for all $N,\,K \in \mathbb{N}$ after an appropriate choice of $x^{N,K}$. From $f(s,\xi,v) \geqslant f^{(qc)}(s,\xi,v)$, we obtain $F(x,u) \geqslant F^{(qc)}(x,u)$ for all feasible (x,u), thus it holds that $m_1 \geqslant m_2$. Summing up, we arrive at $m_1 = m_2 = m_3$.

• **Step 4.** *Completion of the proof.* If $(\hat{x}, \hat{u})$ is a global minimizer of $(\mathrm{P})_1$ then [Wagner 11b], p. 193, Theorem 1.4., implies that $(\hat{x}, \hat{u})$ is a global minimizer of $(\mathrm{P})_2$ as well, and vice versa. Now the proof of Theorem 1.1. will be completed by the observation that for every global minimizer $(\hat{x}, \hat{u})$ of $(\mathrm{P})_2$, it follows that

$$\begin{aligned} m_2 = F^{(qc)}(\hat{x}, \hat{u}) = F(\hat{x}, \hat{u}) = \int_\Omega f(s, \hat{x}(s), \hat{u}(s))\,ds &= \int_\Omega \int_{\mathrm{K}} f(s, \hat{x}(s), v)\,d\delta_{\hat{u}(s)}(v)\,ds \\ &= \widetilde{F}(\hat{x}, \{\delta_{\hat{u}(s)}\}) = m_3\,, \end{aligned} \tag{4.16}$$

and $(\hat{x}, \{\delta_{J\hat{u}(s)}\})$ is a global minimizer of $(\mathrm{P})_3$. ■

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