Quasiconvex relaxation of multidimensional control problems

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1. Introduction.

a) Relaxation of multidimensional variational and optimal control problems.

Relaxation of a given variational or control problem means to define a new problem with the same minimal value as before, whose admissible domain contains the original one (eventually in the sense of an embedding), and whose objective is lower semicontinuous with respect to an appropriate topology. [01] Since the relaxed problem admits global minimizers, it can be accessed by direct methods. [02] For the relaxation of the basic problem of multidimensional calculus of variations,

$$(\mathrm{V})_0 \quad F(x) = \int_\Omega r(t, x(t), Jx(t))\, dt \longrightarrow \inf!\,; \quad x \in W_0^{1,p}(\Omega, \mathbb{R}^n)\,, \quad \Omega \subset \mathbb{R}^m\,, \tag{1.1}$$

two different approaches may be pursued. In the first one, the integrand $r(t, \xi, v)$ within the objective has to be replaced by its convex ($n = 1$) resp. quasiconvex ($n \geqslant 2$) envelope with respect to v. [03] The second approach is the introduction of generalized controls ("Young measures") $\boldsymbol{\mu}\colon \Omega \to rca^{\,pr}(\mathbb{R}^{nm})$ [04] and the replacement of the objective in (1.1) by

$$\widetilde{F}(x, \boldsymbol{\mu}) = \int_\Omega \int_{\mathbb{R}^{nm}} r(\,t, x(t), v\,)\, d\mu_t(v)\, dt \quad \text{with} \quad \frac{\partial x_i}{\partial t_j}(t) = \int_{\mathbb{R}^{nm}} v_{ij}\, d\mu_t(v) \quad \forall\, i, j \quad (\forall)\, t \in \Omega\,.^{05)} \tag{1.2}$$

Both approaches are connected by Jensen's integral inequality for the convex resp. quasiconvex envelope. [06] For multidimensional control problems of Dieudonné-Rashevsky type,

$$(\mathrm{P})_0 \quad F(x, u) = \int_\Omega f(t, x(t), u(t))\, dt \longrightarrow \inf!\,; \quad (x, u) \in W_0^{1,p}(\Omega, \mathbb{R}^n) \times L^p(\Omega, \mathbb{R}^{nm})\,; \tag{1.3}$$

$$Jx(t) = \begin{pmatrix} \dfrac{\partial x_1}{\partial t_1}(t) & \dots & \dfrac{\partial x_1}{\partial t_m}(t) \\ \vdots & & \vdots \\ \dfrac{\partial x_n}{\partial t_1}(t) & \dots & \dfrac{\partial x_n}{\partial t_m}(t) \end{pmatrix} = u(t) \in \mathrm{K} \subset \mathbb{R}^{n \times m} \ \ (\forall)\, t \in \Omega\,; \tag{1.4}$$

$$u \in \mathrm{U} = \big\{\, u \in L^p(\Omega, \mathbb{R}^{nm}) \,\big|\, u(t) \in \mathrm{K} \ \ (\forall)\, t \in \Omega \,\big\} \tag{1.5}$$

with a compact control set $\mathrm{K} \subset \mathbb{R}^{nm}$; however, exclusively the case $n = 1$ has been investigated till now. The main result is the relaxation theorem of Ekeland/Témam (cited as Theorem 1.2. below). Extensions of this theorem with control restrictions of the shape $u \in \mathrm{U} = \{\, u \in L^p(\Omega, \mathbb{R}^{nm}) \,|\, u(t) \in \mathrm{K}(t) \ \ (\forall)\, t \in \Omega \,\}$

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01) Cf. [Buttazzo 89], pp. 2 ff. and pp. 16 ff., [Roubíček 97], pp. vii ff.

02) Cf. [Morrey 66], pp. 15 ff., and [Dacorogna 89], pp. 4 ff.

03) [Dacorogna 89], pp. 228 ff., Theorem 2.1., and pp. 235 ff., Corollaries 2.2. and 2.3.

04) Cf. [Wagner 06b], p. 49, Definition 4.3.

05) [Pedregal 97], pp. 65 ff., particularly Theorem 4.4.

06) Cf. [Kinderlehrer/Pedregal 91], p. 345, Theorem 5.1., [Pedregal 94], p. 65, Proposition 4.2., [Pedregal 97], p. 150, Theorem 8.14., and p. 153, Theorem 8.16., and [Wagner 06b], pp. 130 ff.

have been proved by De Arcangelis et al. but also do not exceed the case $n = 1$. [07] In previous papers of Pickenhain/Wagner, when necessary optimality conditions for problems $(\mathrm{P})_0$ and their Young measure relaxations have been derived for the case $n \geqslant 2$, [08] it was explicitly assumed that the integrand within the objective can be replaced by its convex envelope without a change of minimal value. However, for every $n \geqslant 2$ one can define a problem $(\mathrm{P})_0$ where the minimal value is decreased by this replacement. [09]
It is a well-known fact from the relaxation theory of multidimensional variational problems that the investigation of general integrands $r(t, \xi, v)$ can be reduced to the special case where the integrand depends on v only. [10] For this reason, in the present paper we confine ourselves to the investigation of a model problem with an integrand $f(v)$ and pursue the approach of the replacement of f by an appropriate envelope which turns out to be the *lower semicontinuous quasiconvex envelope* $f^{(qc)}$ (see Definition 2.9. below) instead of the convex envelope. The extension of our result to integrands $r(t, \xi, v)$ will be given in a subsequent paper. [11]
Thus we formulate the following multidimensional control problem:

$$(\mathrm{P}) \quad F(x) = \int_\Omega f(\, Jx(t)\,)\, dt \longrightarrow \inf!\,; \quad x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,; \quad Jx(t) \in \mathrm{K} \;\; (\forall)\, t \in \Omega\,. \tag{1.6}$$

Here we choose $n \geqslant 1$, $m \geqslant 2$, $\Omega \subset \mathbb{R}^m$ as closure of a bounded Lipschitz domain (in strong sense) with $\mathfrak{o} \in \operatorname{int}(\Omega)$, a convex body $\mathrm{K} \subset \mathbb{R}^{nm}$ with $\mathfrak{o} \in \operatorname{int}(\mathrm{K})$ and an integrand $f \in \mathcal{F}_\mathrm{K}$ (cf. Definition 1.4. below) whose restriction to K is continuous.

The following theorem summarizes the procedure of relaxation of (P) by replacement of the integrand:

Theorem 1.1. (Relaxation of the model problem (P) **)** *Consider the problem* (P) *under the assumptions mentioned after* (1.6) *and a function* $f^\#(v)\colon\, \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ *with the following properties:*

1) The set $\operatorname{dom}(f^\#)$ *is Borel measurable with* $K \subseteq \operatorname{dom}(f^\#)$, $f^\# \,\big|\, \operatorname{dom}(f^\#)$ *is a Borel measurable function which is bounded from below on every bounded subset of* $\operatorname{dom}(f^\#)$.

2) It holds that $f^\#(v) \leqslant f(v)$ *for all* $v \in \mathrm{K}$, *consequently* $F^\#(x) = \int_\Omega f^\#(\, Jx(t)\,)\, dt \leqslant \int_\Omega f(\, Jx(t)\,)\, dt = F(x)$ *for all admissible functions in* (P).

3) For every sequence $\{\, x^N \,\}$ *of admissible functions in* (P) *with* $x^N \xrightarrow{\;*\;}_{L^\infty(\Omega,\mathbb{R}^n)} \hat{x}$ *and* $Jx^N \xrightarrow{\;*\;}_{L^\infty(\Omega,\mathbb{R}^{nm})} J\hat{x}$, *the lower semicontinuity relation* $F^\#(\hat{x}) \leqslant \liminf_{N\to\infty} F^\#(x^N)$ *is satisfied.*

4) The minimal values of (P) *and of the following problem* $(\mathrm{P})^\#$ *coincide:*

$$(\mathrm{P})^\# \quad F^\#(x) = \int_\Omega f^\#(\, Jx(t)\,)\, dt \longrightarrow \inf!\,; \quad x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,; \quad Jx(t) \in \mathrm{K} \;\; (\forall)\, t \in \Omega\,. \tag{1.7}$$

Then the (finite) minimal values of problems (P) *and* $(\mathrm{P})^\#$ *are identical, and every minimizing sequence* $\{\, x^N \,\}$ *of* (P) *contains a subsequence* $\{\, x^{N'} \,\}$ *converging weakly** *(in the sense of* $L^\infty(\Omega, \mathbb{R}^n)$ *resp.* $L^\infty(\Omega, \mathbb{R}^{nm})$ *) together with their generalized derivatives to a global minimizer* $\hat{x}$ *of* $(\mathrm{P})^\#$.

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[07] See e. g. [De Arcangelis/Monsurrò/Zappale 04], p. 386, Theorem 6.6., [De Arcangelis/Zappale 05], pp. 267 ff. Section 5. [Carbone/De Arcangelis 02] presents an equally inaccessible exposition.

[08] [Pickenhain/Wagner 00a] , [Pickenhain/Wagner 00b] and [Pickenhain/Wagner 05] .

[09] [Pickenhain 91] , pp. 20 – 23, Example 2.

[10] [Dacorogna 89] , pp. 157 ff. and 228 ff.

[11] Cf. [Dacorogna 89] , pp. 166 ff. and 235 ff.

b) Main result about relaxation of (P)**.**

The case $n = 1$ is covered by the well-known relaxation theorem of Ekeland/Témam:

Theorem 1.2. (Relaxation of the model problem (P) **in special cases,** $n = 1$**)** [12] *Consider* (P) *under the following additional assumptions:* $m \geqslant 2$*,* $n = 1$*, and* $\mathrm{K} = \mathrm{K}(\mathfrak{o}, \varrho) \subset \mathbb{R}^{nm}$ *is a closed ball centered in the origin. The the convex envelope* $f^{\#} = f^c$ *of* f *admits the properties 1) – 4) from Theorem 1.1. Consequently, the problem*

$$(\mathrm{P})^c \quad F^c(x) = \int_\Omega f^c(\, Jx(t)\,)\, dt \longrightarrow \inf!\,; \quad x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,; \quad Jx(t) \in \mathrm{K}\ \ (\forall)\, t \in \Omega \tag{1.8}$$

has the same finite minimal value as problem (P)*, and every minimizing sequence* $\{\, x^N \,\}$ *of* (P) *contains a subsequence* $\{\, x^{N'} \,\}$ *converging weakly* (in the sense of* $L^\infty(\Omega, \mathbb{R}^n)$ *resp.* $L^\infty(\Omega, \mathbb{R}^{nm})$ *) together with their generalized derivatives to a global minimizer* $\hat{x}$ *of* $(\mathrm{P})^c$*.*

In order to extend Ekeland/Témam's theorem to the case $n > 1$, we introduce quasiconvex functions which are allowed to take the value $(+\infty)$ (cf. Definition 2.5. below). Then as the main result of the present paper, the following generalization of Theorem 1.2. can be stated:

Theorem 1.3. (Relaxation of the model problem (P)**,** $n \geqslant 1$**)** *Consider the model problem* (P) *under the assumptions mentioned after* (1.6)*. In particular, we assume* $m \geqslant 2$*,* $n \geqslant 1$*, and* $\mathrm{K} \subset \mathbb{R}^{nm}$ *is an arbitrary convex body with* $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$*. Then the lower semicontinuous quasiconvex envelope* $f^{\#} = f^{(qc)}$ *of* f *(cf. Definition 2.9. and Theorem 2.13. below) admits the properties 1) – 4) from Theorem 1.1. Consequently, the problem*

$$(\mathrm{P})^{(qc)} \quad F^{(qc)}(x) = \int_\Omega f^{(qc)}(\, Jx(t)\,)\, dt \longrightarrow \inf!\,; \quad x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,; \quad Jx(t) \in \mathrm{K}\ \ (\forall)\, t \in \Omega \tag{1.9}$$

has the same finite minimal value as problem (P)*, and every minimizing sequence* $\{\, x^N \,\}$ *of* (P) *contains a subsequence* $\{\, x^{N'} \,\}$ *converging weakly* (in the sense of* $L^\infty(\Omega, \mathbb{R}^n)$ *resp.* $L^\infty(\Omega, \mathbb{R}^{nm})$ *) together with their generalized derivatives to a global minimizer* $\hat{x}$ *of* $(\mathrm{P})^{(qc)}$*.*

Here the envelope $f^{(qc)}\colon\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is defined by (1.10)

$$f^{(qc)}(v) = \sup \big\{\, g(v) \ \big|\ g\colon\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}\ \text{ quasiconvex and lower semicontinuous, } g(v) \leqslant f(v)\ \forall\, v \in \mathbb{R}^{nm} \,\big\}$$

(see Section 2.3.).
Let us remark that in the case $n = 1$, $f^{(qc)}$ and f^c coincide (Theorem 2.14., 3)). Consequently, Theorem 1.3. generalizes Theorem 1.2. in the case $n = 1$ as well, namely, by comprehension of convex bodies K of arbitrary shape.

c) Outline of the paper.

After a short recall of generalized notions of convexity, we consider in *Section 2.a)* quasiconvex functions with values in $\overline{\mathbb{R}} = \mathbb{R} \cup \{\, +\infty \,\}$ and a convex effective domain $\mathrm{K} = \mathrm{dom}\,(f)$. Then we collect the most important properties of the envelope f^* proposed by Dacorogna/Marcellini *(Section 2.b))* and the lower semicontinuous quasiconvex envelope $f^{(qc)}$ *(Section 2.c)).* Now we are positioned to prove the Theorems 1.1. and 1.3. *(Section 3).*

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[12] [Ekeland/Témam 99], p. 327, Corollary 2.17., together with p. 334, Proposition 3.4., and p. 335 f., Proposition 3.6. Loc. cit. the theorem has been formulated for integrand of the shape $f(t, v)$.

d) Notations and abbreviations.

Let $k \in \{0, 1, \dots, \infty\}$ and $1 \leqslant p \leqslant \infty$. Then $C^k(\Omega, \mathbb{R}^r)$, $L^p(\Omega, \mathbb{R}^r)$ and $W^{k,p}(\Omega, \mathbb{R}^r)$ denote the spaces of r-dimensional vector functions whose components are k-times continuously differentiable, resp. belong to $L^p(\Omega)$ or to the Sobolev space of $L^p(\Omega)$-functions with weak derivatives up to kth order in $L^p(\Omega)$ respectively. In addition, functions within the subspaces $C_0^k(\Omega, \mathbb{R}^r) \subset C^k(\Omega, \mathbb{R}^r)$ resp. $W_0^{k,p}(\Omega, \mathbb{R}^r) \subset W_p^{k,p}(\Omega, \mathbb{R}^r)$ are compactly supported. The symbols x_{t_j} and $\partial x / \partial t_j$ may denote the classical as well as the weak partial derivative of x by t_j.
We denote by $\operatorname{int}(\mathrm{A})$, $\operatorname{ri}(\mathrm{A})$, $\partial(\mathrm{A})$, $\operatorname{rb}(\mathrm{A})$, $\operatorname{cl}(\mathrm{A})$, $\operatorname{co}(\mathrm{A})$ and $|\,\mathrm{A}\,|$ the interior, relative interior, boundary, relative boundary, closure, the convex hull and the r-dimensional Lebesgue measure of a set $\mathrm{A} \subseteq \mathbb{R}^r$, respectively. The characteristic function of the set $\mathrm{A} \subseteq \mathbb{R}^r$ is defined as $\mathbb{1}_{\mathrm{A}} : \mathbb{R}^r \to \mathbb{R}$ with $\mathbb{1}_{\mathrm{A}}(t) = 1 \iff t \in \mathrm{A}$ and $\mathbb{1}_{\mathrm{A}}(t) = 0 \iff t \notin \mathrm{A}$. We set $\overline{\mathbb{R}} = \mathbb{R} \cup \{(+\infty)\}$ and equip $\overline{\mathbb{R}}$ with the natural topological and order structures where $(+\infty)$ is the greatest element. Throughout the whole paper, we consider only *proper functions* $f : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$, assuming that $\operatorname{dom}(f) = \{ v \in \mathbb{R}^{nm} \mid f(v) < (+\infty) \}$ is always nonempty. The restriction of a function f to the subset A of its range of definition is denoted by $f \,\big|\, \mathrm{A}$.

Definition 1.4. (Function class $\mathcal{F}_{\mathrm{K}}$) *Let $\mathrm{K} \subset \mathbb{R}^{nm}$ be a given convex body with $\mathfrak{o} \in \operatorname{int}(\mathrm{K})$. We say that a function $f : \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ belongs to the class $\mathcal{F}_{\mathrm{K}}$ iff $f \,\big|\, \mathrm{K} \in C^0(\mathrm{K}, \mathbb{R})$ and $f \,\big|\, (\mathbb{R}^{nm} \setminus \mathrm{K}) \equiv (+\infty)$.*

Consequently, any function $f \in \mathcal{F}_{\mathrm{K}}$ is bounded and uniformly continuous on K, and the class $\mathcal{F}_{\mathrm{K}}$ and the Banach space $C^0(\mathrm{K}, \mathbb{R})$ are isomorphic and isometric.
A convex body $\mathrm{K} \subset \mathbb{R}^{nm}$ will be understood as a convex compact set with nonempty interior.[13] A point $v \in \mathrm{K}$ is called extremal point of K iff from $v = \lambda' v' + \lambda'' v''$, $\lambda', \lambda'' > 0$, $\lambda' + \lambda'' = 1$, $v', v'' \in \mathrm{K}$ it follows $v' = v'' = v$. The set of all extremal points of K is denoted by $\operatorname{ext}(\mathrm{K})$. For a convex body, $\operatorname{ext}(\mathrm{K})$ is always nonempty. A convex subset $\Phi \subseteq \mathrm{K}$ is called a face of K iff from $v \in \Phi$ and $v = \lambda' v' + \lambda'' v''$, $\lambda', \lambda'' > 0$, $\lambda' + \lambda'' = 1$, $v', v'' \in \mathrm{K}$ it follows: $v', v'' \in \Phi$. K itself as well as $\emptyset$ will be regarded as improper faces. All (nonempty) faces of a convex body are compact sets. The dimension k of a face is the dimension of its affine hull; we define $\operatorname{Dim}(\emptyset) = (-1)$. Thus the null-dimensional faces of K are precisely the singletons $\{x\}$, $x \in \operatorname{ext}(\mathrm{K})$.
We close this subsection with *three nonstandard notations.* "$\{x^N\}$, A" denotes a sequence $\{x^N\}$ with members $x^N \in \mathrm{A}$. If $\mathrm{A} \subseteq \mathbb{R}^r$ then the abbreviation "$(\forall)\, t \in \mathrm{A}$" has to be read as "for almost all $t \in \mathrm{A}$" resp. "for all $t \in \mathrm{A}$ except a r-dimensional Lebesgue null set". The symbol $\mathfrak{o}$ denotes, depending on the context, the zero element resp. the zero function of the underlying space.

2. The lower semicontinuous quasiconvex envelope.

a) Generalized convexity notions.

Definition 2.1. (Convexity notions for functions with values in $\mathbb{R}$)[14]

1) **(Convex function)** *A function $f : \mathbb{R}^{nm} \to \mathbb{R}$ is said to be convex if Jensen's inequality is satisfied for every $v', v'' \in \mathbb{R}^{nm}$:*

$$f(\lambda' v' + \lambda'' v'') \leqslant \lambda' f(v') + \lambda'' f(v'') \quad \forall \lambda', \lambda'' \geqslant 0,\ \lambda' + \lambda'' = 1. \tag{2.1}$$

[13] We follow [Brøndsted 83] and [Schneider 93].

[14] In the present paper, the concept of polyconvexity will not be used.

2) **(Quasiconvex function)** [15] *A function $f:\ \mathbb{R}^{nm}\to\mathbb{R}$ is said to be quasiconvex if it is Borel measurable, bounded from below on every bounded subset of $\mathbb{R}^{nm}$, and satisfies Morrey's integral inequality for all $v\in\mathbb{R}^{nm}$:*

$$f(v)\ \leqslant\ \frac{1}{|\,\Omega\,|}\int_\Omega f(\,v+Jx(t)\,)\,dt\quad\forall\,x\in W_0^{1,\infty}(\Omega,\mathbb{R}^n)\,;\tag{2.2}$$

or equivalently

$$f(v)\ =\ \inf\Big\{\,\frac{1}{|\,\Omega\,|}\int_\Omega f(\,v+Jx(t)\,)\,dt\ \Big|\ x\in W_0^{1,\infty}(\Omega,\mathbb{R}^n)\,\Big\}\,.\tag{2.3}$$

Here $\Omega\subset\mathbb{R}^m$ is the closure of a bounded Lipschitz domain (in strong sense).

3) **(Rank one convex function)** *A function $f:\ \mathbb{R}^{nm}\to\overline{\mathbb{R}}$ is said to be rank one convex if Jensen's inequality is satisfied in any rank one direction: for every v', $v''\in\mathbb{R}^{nm}$ (considered as (n,m)-matrices) it holds:*

$$\operatorname{Rank}(v'-v'')\leqslant 1\ \Longrightarrow\ f(\,\lambda'\,v'+\lambda''\,v''\,)\ \leqslant\ \lambda'\,f(v')+\lambda''\,f(v'')\quad\forall\,\lambda',\,\lambda''\geqslant 0\,,\ \lambda'+\lambda''=1\,.\tag{2.4}$$

These properties have in common that they are conserved under the forming of a pointwise supremum. Thus the following generalized convex envelopes are well-defined:

Definition 2.2. (Generalized convex envelopes) *Let $f:\ \mathbb{R}^{nm}\to\mathbb{R}$ be a function bounded from below.*

1) **(Convex envelope f^c)** *The convex envelope $f^c:\ \mathbb{R}^{nm}\to\mathbb{R}$ of f is defined by*

$$f^c(v)\ =\ \sup\big\{\,g(v)\ \big|\ g:\ \mathbb{R}^{nm}\to\mathbb{R}\quad\textit{convex},\ \ g(v)\leqslant f(v)\ \ \forall\,v\in\mathbb{R}^{nm}\,\big\}\,.\tag{2.5}$$

2) **(Quasiconvex envelope f^{qc})** *The quasiconvex envelope $f^{qc}:\ \mathbb{R}^{nm}\to\mathbb{R}$ of f is defined by*

$$f^{qc}(v)\ =\ \sup\big\{\,g(v)\ \big|\ g:\ \mathbb{R}^{nm}\to\mathbb{R}\quad\textit{quasiconvex},\ \ g(v)\leqslant f(v)\ \ \forall\,v\in\mathbb{R}^{nm}\,\big\}\,.\tag{2.6}$$

3) **(Rank one convex envelope f^{rc})** *The rank one convex envelope $f^{rc}:\ \mathbb{R}^{nm}\to\mathbb{R}$ of f is defined by*

$$f^{rc}(v)\ =\ \sup\big\{\,g(v)\ \big|\ g:\ \mathbb{R}^{nm}\to\mathbb{R}\quad\textit{rank one convex},\ \ g(v)\leqslant f(v)\ \ \forall\,v\in\mathbb{R}^{nm}\,\big\}\,.\tag{2.7}$$

Theorem 2.3. (Relations between the generalized convexity notions) [16]

1) For any function $f:\ \mathbb{R}^{nm}\to\mathbb{R}$, we have the implications: f convex $\Longrightarrow$ f quasiconvex $\Longrightarrow$ f rank one convex $\Longrightarrow$ f separately convex. If $n=1$ or $m=1$ then we have the equivalences f convex $\Longleftrightarrow$ f quasiconvex $\Longleftrightarrow$ f rank one convex.

2) For any function $f:\ \mathbb{R}^{nm}\to\mathbb{R}$ bounded from below, the following inequalities hold:

$$f^c(v)\ \leqslant\ f^{qc}(v)\ \leqslant\ f^{rc}(v)\ \leqslant\ f(v)\quad\forall\,v\in\mathbb{R}^{nm}\ .\tag{2.8}$$

For the quasiconvex envelope, Dacorogna's representation theorem holds:

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[15] [Dacorogna 89] , p. 99, Definition ii). The original definition has been slightly changed in order to guarantee the integrability of all compositions $f(\,v+Jx(\,\cdot\,)\,)$.

[16] [Dacorogna 89] , p. 102, Theorem 1.1., i) and ii).

Theorem 2.4. (Representation of f^{qc} for functions with values in $\mathbb{R}$) [17] *Assume that $f:\ \mathbb{R}^{nm} \to \mathbb{R}$ is Borel measurable, bounded from below on $\mathbb{R}^{nm}$ and bounded from above on every compact subset of $\mathbb{R}^{nm}$. Then $f^{qc}(v)$ admits the representation*

$$f^{qc}(v) \;=\; \inf \Big\{ \frac{1}{|\,\Omega\,|} \int_\Omega f(\,v + Jx(t)\,)\,dt \;\Big|\; x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,\Big\} \tag{2.9}$$

for all $v \in \mathbb{R}^{nm}$ where $\Omega \subset \mathbb{R}^m$ is the closure of a bounded Lipschitz domain (in strong sense).

While the notions of convexity as well as of rank one convexity can be applied to functions $f:\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ as well, it turned out to be much more difficult to extend the notion of quasiconvexity to functions with values in $\overline{\mathbb{R}}$ in a reasonable way. [18] Following [Wagner 06a] – [Wagner 06c] , we adopt the following extension of Definition 2.1., 2):

Definition 2.5. (Quasiconvex functions with values in $\overline{\mathbb{R}}$) [19] *A function $f:\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ with the following properties is said to be quasiconvex:*

1) $\mathrm{dom}\,(f) \subseteq \mathbb{R}^{nm}$ *is a (nonempty) Borel set;*

2) $f\,\big|\,\mathrm{dom}\,(f)$ *is Borel measurable and bounded from below on every bounded subset of* $\mathrm{dom}\,(f)$*;*

3) for all $v \in \mathbb{R}^{nm}$, f satisfies Morrey's integral inequality (2.2) *resp.* (2.3).

We agree with the convention that the integral $\int_{\mathrm{A}} (+\infty)\,dt$ takes the values zero or $(+\infty)$ if either $\mathrm{A} \subseteq \mathbb{R}^m$ is an m-dimensional Lebesgue null set or has positive measure. Note that, in Morrey's integral inequality, the values of the integrand f cannot be changed even on a Lebesgue null set of $\mathbb{R}^{nm}$. The appropriate extension of the definition of the quasiconvex envelope will be given in Section 2.c) below.

b) The envelope f^* related to K.

In this subsection, we present Dacorogna/Marcellini's idea to define a "quasiconvex" envelope f^*, which is adapted to the control restriction in (P), by introducing the restriction $v + Jx(t) \in \mathrm{K}\ (\forall)\,t \in \Omega$ into the representation formula (2.9) for f^{qc}. [20] Let us fix a convex body $\mathrm{K} \subset \mathbb{R}^{nm}$ with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$ and the quantities $c_{\mathrm{K}} = \mathrm{Dist}\,(\mathfrak{o}\,,\,\partial\mathrm{K})$ and $C_{\mathrm{K}} = \mathrm{Max}\,\big(\,1\,,\ \mathrm{Max}_{\,v\in\mathrm{K}}\ |\,v\,|\,\big)$, thus $0 < c_{\mathrm{K}} \leqslant C_{\mathrm{K}}$ and $\mathrm{Diam}\,(\mathrm{K}) \leqslant 2\,C_{\mathrm{K}}$.

Definition 2.6. (Envelope f^* related to K) *Let $f:\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ be a function with the following properties: the set* $\mathrm{dom}\,(f)$ *is measurable,* $f\,\big|\,\mathrm{dom}\,(f)$ *is a measurable function, and f is bounded from below on $\mathbb{R}^{nm}$. $\mathrm{K} \subset \mathbb{R}^{nm}$ is the convex body mentioned above. Then we define for $v \in \mathbb{R}^{nm}$:*

$$f^*(v) \;=\; \inf \Big\{ \frac{1}{|\,\Omega\,|} \int_\Omega f(\,v + Jx(t)\,)\,dt \;\Big|\; x \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,,\ v + Jx(t) \in \mathrm{K}\ \ (\forall)\,t \in \Omega\,\Big\} \in \overline{\mathbb{R}}\,. \tag{2.10}$$

The function f^* has been introduced in [Kinderlehrer/Pedregal 91] , p. 356, in the special case $\mathrm{K} = \mathrm{K}(\mathfrak{o}, \varrho)$ and in [Dacorogna/Marcellini 97] , p. 27, Theorem 7.2., for arbitrary convex bodies K. In contrast to both these papers where $f \in C^0(\mathrm{K}\,,\,\mathbb{R})$ was assumed, we formulate the definition from the outset

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[17] [Dacorogna 89] , p. 201, Theorem 1.1., (4); first proven in [Dacorogna 82] , p. 108, Theorem 5, in a special case.

[18] By the author's knowledge, such functions have been considered up to now only in [Ball/Murat 84] , [Dacorogna/ Fusco 85] , [Kinderlehrer/Pedregal 91] , [Wagner 06a] , [Wagner 06b] and [Wagner 06c] .

[19] [Wagner 06a] , p. 237, Definition 5, as specification of [Ball/Murat 84] , p. 228, Definition 2.1., in the case $p = \infty$. The definition has been changed in the same way as Definition 2.1., 2).

[20] [Dacorogna/Marcellini 97] , p. 27.

for functions $f:\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$. The main property of f^* is the following continuity relation depending not only on the distance of two given points $v',\, v'' \in \varPhi$ but also on their distances to the relative boundary $\mathrm{rb}\,(\varPhi)$ of the face $\varPhi \subset \mathrm{K}$:

Theorem 2.7. (ε-δ relation for the restriction of f^* to faces of K**)** [21] *Let $f \in \mathcal{F}_\mathrm{K}$ and a k-dimensional face $\varPhi \subseteq \mathrm{K}$, $0 \leqslant k \leqslant nm$, be given. Assume that the uniform continuity of f on* K *is described through the ε-δ relation*

$$\big|\, v' - v'' \,\big| \;\leqslant\; \delta(\varepsilon) \;<\; 1 \;\Longrightarrow\; \big|\, f(v') - f(v'') \,\big| \leqslant \varepsilon \quad \forall\, v',\, v'' \in \mathrm{K}\,. \tag{2.11}$$

Then $f^* \,\big|\, \varPhi$ obeys the following ε-δ relation:

$$\begin{aligned} \big|\, v' - v'' \,\big| \;\leqslant\; \delta_1(\varepsilon) \,\cdot\, \mathrm{Min}\,\big(\, 1 \,,\, \mathrm{Dist}\,(v',\, \mathrm{rb}\,(\varPhi)) \,,\, \mathrm{Dist}\,(v'',\, \mathrm{rb}\,(\varPhi))\,\big) \;\Longrightarrow& \\ \big|\, f^*(v') - f^*(v'') \,\big| \;\leqslant\; 2\,\varepsilon \quad \forall\, v',\, v'' \in \mathrm{ri}\,(\varPhi)& \end{aligned} \tag{2.12}$$

with $\delta_1(\varepsilon) = \frac{1}{4}\,\delta(\varepsilon)/C_\mathrm{K}$ where C_K is the quantity defined in the beginning of the subsection.

Due to this theorem, all restrictions $f^* \,\big|\, \mathrm{ri}\,(\varPhi)$ are continuous. A further important consequence is the radial continuity of f^*:

Theorem 2.8. [22] *Let $f \in \mathcal{F}_\mathrm{K}$ be given.*

1) **(ε-δ relation for f^* along rays starting from the origin)** *Assume that the uniform continuity of f on* K *is described through the ε-δ relation*

$$\big|\, v' - v'' \,\big| \;\leqslant\; \delta(\varepsilon) \;<\; 1 \;\Longrightarrow\; \big|\, f(v') - f(v'') \,\big| \leqslant \varepsilon \quad \forall\, v',\, v'' \in \mathrm{K}\,. \tag{2.13}$$

Assume further that two points $v,\, w \in \mathrm{int}\,(\mathrm{K})$ admit the following properties: a) $v,\, w$ are situated on the same ray R *starting from $\mathfrak{o}$, and b) $0 < \mathrm{Dist}\,(w\,,\, \partial\mathrm{K}) < \mathrm{Dist}\,(v\,,\, \partial\mathrm{K}) < \frac{1}{2}\, c_\mathrm{K}$. Then f^* obeys the following ε-δ estimate, which holds uniformly for all rays* R *starting from $\mathfrak{o}$:*

$$\mathrm{Dist}\,(w\,,\, v) \;\leqslant\; \delta_2(\varepsilon) \;\Longrightarrow\; f^*(w) - f^*(v) \;\geqslant\; -2\,\varepsilon \tag{2.14}$$

with $\delta_2(\varepsilon) = \frac{1}{6}\,\delta(\varepsilon) \cdot c_\mathrm{K}/C_\mathrm{K}$ where c_K and C_K are the quantities defined in the beginning of the subsection.

2) **(Existence of the radial limit of f^* along rays starting from the origin)** *Along every ray* R *starting from the origin, the following limit in the point $v_0 \in \mathrm{R} \cap \partial\mathrm{K}$ exists:*

$$\lim_{v \to v_0\,,\, v \,\in\, \mathrm{R} \,\cap\, \mathrm{int}\,(\mathrm{K})} f^*(v)\,. \tag{2.15}$$

3) **(ε-δ relation for f^* extended by its radial limits along rays starting from the origin)** *Under the assumptions of Part 1), we consider two points $v,\, w \in \mathrm{K}$, which a) are situated on the same ray* R *starting from $\mathfrak{o}$, and b) satisfy $0 \leqslant \mathrm{Dist}\,(w\,,\, \partial\mathrm{K}) \leqslant \mathrm{Dist}\,(v\,,\, \partial\mathrm{K}) < \frac{1}{2}\, c_\mathrm{K}$. Then the ε-δ estimate from Part 1) can be extended to the function $f^{\#}:\ \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ defined through*

$$f^{\#}(v_0) = \begin{cases} \qquad\qquad f^*(v_0) \;\big|\; v_0 \in \mathrm{int}\,(\mathrm{K})\,; \\ \displaystyle\lim_{v \to v_0\,,\, v \,\in\, \mathrm{R} \,\cap\, \mathrm{int}\,(\mathrm{K})} f^*(v) \;\big|\; v_0 \in \partial\mathrm{K}\,; \\ \qquad\qquad (+\infty) \;\big|\; v_0 \in \mathbb{R}^{nm} \setminus \mathrm{K}\,. \end{cases} \tag{2.16}$$

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[21] [Wagner 06b] , p. 23, Theorem 3.5., resp. [Wagner 06c] , p. 16, Theorem 3.5.

[22] [Wagner 06b] , p. 29, Theorem 3.12., resp. [Wagner 06c] , p. 22, Theorem 3.12.

Namely, the estimate

$$\operatorname{Dist}(w\,,\,v)\;\leqslant\;\delta_2(\varepsilon)\quad\Longrightarrow\quad f^{\#}(w)-f^{\#}(v)\;\geqslant\;-2\,\varepsilon\,, \tag{2.17}$$

holds uniformly for all rays R *starting from* $\mathfrak{o}$*.*

c) The lower semicontinuous quasiconvex envelope $f^{(qc)}$ and its representation.

Definition 2.9. (Lower semicontinuous quasiconvex envelope $f^{(qc)}$ for functions with values in $\overline{\mathbb{R}}$) [23] *To any function $f\colon\ \mathbb{R}^{nm}\to\overline{\mathbb{R}}$ bounded from below, we define the lower semicontinuous quasiconvex envelope*

$$f^{(qc)}(v)\;=\;\sup\big\{\,g(v)\;\big|\;g\colon\ \mathbb{R}^{nm}\to\overline{\mathbb{R}}\ \text{ quasiconvex and lower semicontinuous,}\\ g(v)\leqslant f(v)\ \ \forall\,v\in\mathbb{R}^{nm}\,\big\}\,. \tag{2.18}$$

Remarks. a) Definition 2.9. is motivated by the observation that finite quasiconvex functions $g\colon\ \mathbb{R}^{nm}\to\mathbb{R}$ are from the outset continuous functions. [24] Consequently, if a function f is bounded from below and takes only finite values then the envelopes f^{qc} from Definition 2.2., 2) and $f^{(qc)}$ from Definition 2.9. coincide, and $f^{(qc)}$ is quasiconvex and continuous.

b) For an arbitrary function bounded from below with $\operatorname{dom}(f)\neq\mathbb{R}^{nm}$, however, it is even questionable whether $f^{(qc)}$ satisfies condition 2) from Definition 2.5. Nevertheless, we call $f^{(qc)}$ lower semicontinuous quasiconvex envelope of f in this case as well.

Lemma 2.10. [25] *Assume that the functions f_1 and $f_2\colon\ \mathbb{R}^{nm}\to\overline{\mathbb{R}}$ are bounded from below. Then we have the implication: $f_1(v)\;\leqslant\;f_2(v)\ \forall\,v\in\mathbb{R}^{nm}\implies {f_1}^{(qc)}(v)\;\leqslant\;{f_2}^{(qc)}(v)\ \forall\,v\in\mathbb{R}^{nm}$.*

We emphasize that the following four theorems are formulated for functions $f\in\mathfrak{F}_{\mathrm{K}}$.

Theorem 2.11. (Properties of $f^{(qc)}$ for $f\in\mathfrak{F}_{\mathrm{K}}$) [26] *For any function $f\in\mathfrak{F}_{\mathrm{K}}$ it holds:*

1) $f^c(v)\;\leqslant\;f^{(qc)}(v)\;\leqslant\;f(v)$ for all $v\in\mathbb{R}^{nm}$, which implies particularly $f^{(qc)}(v)=(+\infty)$ for all $v\in\mathbb{R}^{nm}\setminus\mathrm{K}$ and $f^{(qc)}(v)=f(v)$ for all $v\in\operatorname{ext}(\mathrm{K})$.

2) $f^{(qc)}$ is lower semicontinuous and quasiconvex.

3) The restriction $f^{(qc)}\,\big|\operatorname{int}(\mathrm{K})$ is continuous.

According to Part 2) of this theorem, $f^{(qc)}$ is admissible in the process of its own forming. The immediate consequence of this fact is

Theorem 2.12. ($f^{(qc)}$ for $f\in\mathfrak{F}_{\mathrm{K}}$ as the largest quasiconvex, lower semicontinuous function $g\leqslant f$) [27] *Let $f\in\mathfrak{F}_{\mathrm{K}}$. For any lower semicontinuous, quasiconvex function $g\colon\ \mathbb{R}^{nm}\to\overline{\mathbb{R}}$ from $g(v)\leqslant f(v)$ $\forall\,v\in\mathbb{R}^{nm}$ it follows $g(v)\leqslant f^{(qc)}(v)\ \forall\,v\in\mathrm{K}$.*

It turns out that the function $f^{\#}$ defined in Theorem 2.8., 3) and $f^{(qc)}$ are even identical:

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[23] [WAGNER 06B] , p. 14, Definition 2.16., 2), and [WAGNER 06C] , p. 9, Definition 2.14., 2).

[24] [DACOROGNA 89] , p. 29, Theorem 2.3., 2), together with p. 101, Theorem 1.1., i).

[25] [WAGNER 06B] , p. 15, Lemma 2.17., 3), resp. [WAGNER 06C] , p. 10, Lemma 2.15., 3).

[26] [WAGNER 06B] , p. 15, Theorem 2.19., resp. [WAGNER 06C] , p. 10, Theorem 2.17.

[27] [WAGNER 06B] , p. 15, Theorem 2.20., resp. [WAGNER 06C] , p. 10, Theorem 2.18.

Theorem 2.13. (Representation theorem for $f^{(qc)}$) [28] *Let a function $f \in \mathcal{F}_{\mathrm{K}}$ be given. Then the lower semicontinuous quasiconvex envelope $f^{(qc)}\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ admits the representation*

$$f^{(qc)}(v_0) = \begin{cases} f^*(v_0) \;\big|\; v_0 \in \operatorname{int}(\mathrm{K})\,; \\ \lim\limits_{v \to v_0\,,\, v \in \mathrm{R} \cap \operatorname{int}(\mathrm{K})} f^*(v) \;\big|\; v_0 \in \partial \mathrm{K}\,; \\ (+\infty) \;\big|\; v_0 \in \mathbb{R}^{nm} \setminus \mathrm{K}\,. \end{cases} \tag{2.19}$$

Theorem 2.13. generalizes Dacorogna's representation theorem for the quasiconvex envelope (Theorem 2.4.). As a corollary, we infer that $f^{(qc)}$ is rank one convex as well. Moreover, in the special cases $n = 1$ or $m = 1$, the generalized convex envelopes coincide.

Theorem 2.14. *Let a function $f \in \mathcal{F}_{\mathrm{K}}$ be given.*
1) [29] **(Rank one convexity of $f^{(qc)}$)** *Then the function $f^{(qc)}\colon \mathbb{R}^{nm} \to \overline{\mathbb{R}}$ is rank one convex.*
2) **(Inequalities between f^c, $f^{(qc)}$, f^{rc} and f)** *For all $v \in \mathbb{R}^{nm}$, the following inequalities hold:*

$$f^c(v) \leqslant f^{(qc)}(v) \leqslant f^{rc}(v) \leqslant f(v)\,. \tag{2.20}$$

3) [30] **(Coincidence of f^c and $f^{(qc)}$ for $n = 1$ or $m = 1$)** *If $n = 1$ or $m = 1$ then the envelopes f^c, $f^{(qc)}$ and f^{rc} are identical.*

3. Proof of the relaxation theorems.

a) Some assertions from measure theory.

Lemma 3.1. (Strongly Lipschitz domains are squarable) *If $\Omega \subset \mathbb{R}^m$ is the closure of a strongly Lipschitz domain, then $\partial\Omega$ is a m-dimensional Lebesgue null set.*

Proof. We rely upon [Morrey 66], p. 72, Definition 3.4.1.; consequently, $\partial\Omega$ can be covered by open cuboids $\{\, \mathrm{Q}_t \subset \mathbb{R}^m \;\big|\; t \in \partial\Omega \,\}$ such that every $\mathrm{Q}_t \cap \partial\Omega$ is a subset of the graph of a Lipschitz function $f_t\colon \mathbb{R}^{m-1} \to \mathbb{R}$. Since $\partial\Omega$ is compact, we may choose a finite subcovering. Consequently, $\partial\Omega$ can be represented as union of the graphs of finitely many Lipschitz functions with compact ranges of definitions. From [Elstrodt 96], p. 69 f., 7.6. e) and i), it follows that $\partial\Omega$ is a Lebesgue null set. ∎

Theorem 3.2. (Vitali covering theorem) [31] *Consider two sets $\Omega \subset \mathbb{R}^m$ and $\mathrm{G} \subset \mathbb{R}^m$ where G is a compact set of positive Lebesgue measure with $v_0 \in \operatorname{int}(\mathrm{G})$. Let $\mathfrak{G}$ be a family, consisting of sets which have been obtained from G by dilatations with center v_0 and translations. Moreover, assume that $\mathfrak{G}$ has the property*

$(*)$ *For almost all $t \in \operatorname{int}(\Omega)$ and for any $\varepsilon > 0$, there exists a set $\mathrm{G}(t, \varepsilon) \in \mathfrak{G}$ with $t \in \mathrm{G}(t, \varepsilon)$ and* $\operatorname{Diam}(\mathrm{G}(t, \varepsilon)) < \varepsilon$.

The $\mathfrak{G}$ contains an at most countable subfamily $\mathfrak{G}' \subset \mathfrak{G}$ of mutually disjoint sets G_1, G_2, ... $\subseteq \operatorname{int}(\Omega)$ with $\big|\operatorname{int}(\Omega) \setminus \bigcup_{i=1}^{\infty} \mathrm{G}_i\big| = 0$.

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[28] [Wagner 06b], p. 38, Theorem 3.22., resp. [Wagner 06c], p. 29, Theorem 4.1.

[29] Parts 1) and 2): [Wagner 06b], p. 38, Theorem 3.23., resp. [Wagner 06c], p. 29, Theorem 4.2.

[30] [Wagner 06b], p. 38, Theorem 3.24., resp. [Wagner 06c], p. 29, Theorem 4.3.

[31] [Dacorogna/Marcellini 99], p. 231 f., Corollary 10.6.

Theorem 3.3. (Mean value representation of $u \in L^1(\Omega, \mathbb{R}^{nm})$) [32] *For every function $u \in L^1(\Omega, \mathbb{R}^{nm})$ and for almost all points $t \in \Omega$ (the Lebesgue points of u) it holds:*

$$\lim_{\delta \to 0} \frac{1}{|\,\mathrm{Q}(t,\delta)\,|} \int_{\mathrm{Q}(t,\delta)} \big|\, u_{ij}(s) - u_{ij}(t) \,\big|\, ds \;=\; 0\,, \quad 1 \leqslant i \leqslant n\,,\; 1 \leqslant j \leqslant m\,, \tag{3.1}$$

where $\mathrm{Q}(t,\delta)$ denotes the closed cube with center t and edge length $\delta > 0$.

Lemma 3.4. *Given a squarable set $\Omega \subset \mathbb{R}^m$ as the closure of a bounded domain and a function $u \in L^1(\Omega, \mathbb{R}^{nm})$, then for every $N \in \mathbb{N}$ and $\varepsilon > 0$, one can find finitely many closed cubes $\mathrm{Q}(t_s, \delta_s) \subseteq \Omega$, $1 \leqslant s \leqslant r$, with edge length $0 < \delta_s \leqslant 1/N$, with the following properties:*

1) The cubes $\mathrm{Q}(t_s, \delta_s)$ are mutually disjoint;

2) $$\Big|\, \Omega \setminus \bigcup\nolimits_{s=1}^{r} \mathrm{Q}(t_s, \delta_s) \,\Big| \;\leqslant\; \varepsilon\,; \tag{3.2}$$

3) $$\Big|\, u_{ij}(t) - \frac{1}{|\,\mathrm{Q}(t_s,\delta_s)\,|} \int_{\mathrm{Q}(t_s,\delta_s)} u_{ij}(\tau)\, d\tau \,\Big| \;\leqslant\; \varepsilon \quad (\forall)\, t \in \mathrm{Q}(t_s,\delta_s)\,,\; 1 \leqslant s \leqslant r\,,\; 1 \leqslant i \leqslant n\,,\; 1 \leqslant j \leqslant m\,. \tag{3.3}$$

Proof. Let $\Omega_0 \subset \mathrm{int}\,(\Omega)$ be the set of all Lebesgue points of u in the interior of Ω. By Theorem 3.3., we have $|\,\mathrm{int}\,(\Omega) \setminus \Omega_0\,| = 0$, and for every $t \in \Omega_0$ there exists a $\delta(t,\varepsilon) > 0$ such that for all $0 < \delta \leqslant \delta(t,\varepsilon)$ it holds:

$$\mathrm{Q}(t,\delta) \subset \mathrm{int}\,(\Omega) \quad \text{and} \quad \Big|\, u_{ij}(t) - \frac{1}{|\,\mathrm{Q}(t,\delta)\,|} \int_{\mathrm{Q}(t,\delta)} u_{ij}(\tau)\, d\tau \,\Big| \;\leqslant\; \varepsilon \quad (\forall)\, t \in \mathrm{Q}(t,\delta)\,,\; 1 \leqslant i \leqslant n\,,\; 1 \leqslant j \leqslant m\,. \tag{3.4}$$

Consider now the family $\mathfrak{G}$ consisting of all closed cubes $\mathrm{Q}(t,\delta)$ with $t \in \Omega_0$ and $0 < \delta \leqslant \mathrm{Min}\,(1/N\,,\, \delta(t,\varepsilon)\,)$; this family satisfies the assumptions of Theorem 3.2. Consequently, we find an at most countable subfamily of mutually disjoint cubes $\{\, \mathrm{Q}(t_s,\delta_s) \,\}$ with

$$\Big|\, \Omega \setminus \bigcup\nolimits_s \mathrm{Q}(t_s,\delta_s) \,\Big| \;=\; \Big|\, \mathrm{int}\,(\Omega) \setminus \bigcup\nolimits_s \mathrm{Q}(t_s,\delta_s) \,\Big| \;=\; \Big|\, \big\{\, t \in \Omega_0 \;\big|\; t \notin \bigcup\nolimits_s \mathrm{Q}(t_s,\delta_s) \big\} \,\Big| \;=\; 0\,. \tag{3.5}$$

If this family is finite then our assertion is proven, otherwise, we infer the existence of a number $r \in \mathbb{N}$ with

$$\Big|\, \big\{\, t \in \Omega_0 \;\big|\; t \notin \bigcup\nolimits_{s=1}^{r} \mathrm{Q}(t_s,\delta_s) \big\} \,\Big| \;=\; \Big|\, \Omega \setminus \bigcup\nolimits_{s=1}^{r} \mathrm{Q}(t_s,\delta_s) \,\Big| \;\leqslant\; \varepsilon \tag{3.6}$$

from the σ-additivity of the measure. ■

b) Proof of Theorem 1.1.

Proof of Theorem 1.1. The assumptions about (P) guarantee that its minimal value is finite. For every minimizing sequence $\{\, x^N \,\}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ of (P), there exists a subsequence $\{\, x^{N'} \,\}$ with $x^{N'} \xrightarrow{\;*\;}_{L^\infty(\Omega,\mathbb{R}^n)} \hat{x}$, $Jx^{N'} \xrightarrow{\;*\;}_{L^\infty(\Omega,\mathbb{R}^{nm})} J\hat{x}$ and $J\hat{x}(t) \in \mathrm{K}$ $(\forall)\, t \in \Omega$. [33] By 2), we find

$$F^{\#}(x^{N'}) \;\leqslant\; F(x^{N'}) \quad \forall\, N' \in \mathbb{N}, \tag{3.7}$$

and from 3) it follows that

$$F^{\#}(\hat{x}) \;\leqslant\; \liminf_{N' \to \infty} F^{\#}(x^{N'}) \;\leqslant\; \liminf_{N' \to \infty} F(x^{N'}) \;=\; \lim_{N \to \infty} F(x^N) \;=\; m\,. \tag{3.8}$$

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[32] [Evans/Gariepy 92], p. 44, Corollary 1; the balls may be replaced by cubes.

[33] Cf. [Pickenhain/Wagner 00a], p. 223, Lemma 2.3.

Denoting the minimal value of $(\mathrm{P})^{\#}$ by $m^{\#}$, from 4) we get

$$m^{\#} \leqslant F^{\#}(\hat{x}) \leqslant m = m^{\#}\,, \tag{3.9}$$

and $\hat{x}$ is a global minimizer of $(\mathrm{P})^{\#}$. ∎

c) Proof of Theorem 1.3.

As before, let $\mathrm{K} \subset \mathbb{R}^{nm}$ be a fixed convex body with $\mathfrak{o} \in \mathrm{int}\,(\mathrm{K})$ and the quantities $c_{\mathrm{K}} = \mathrm{Dist}\,(\mathfrak{o}\,,\partial\mathrm{K})$ and $C_{\mathrm{K}} = \mathrm{Max}\,\big(1\,, \mathrm{Max}_{v\in\mathrm{K}}\,|\,v\,|\big)$, thus $0 < c_{\mathrm{K}} \leqslant C_{\mathrm{K}}$ and $\mathrm{Diam}\,(\mathrm{K}) \leqslant 2\,C_{\mathrm{K}}$.

Proof of Theorem 1.3. For $f \in \mathcal{F}_{\mathrm{K}}$, $f^{(qc)}$ admits the claimed properties 1) and 2) as a quasiconvex function with $f^{(qc)} \leqslant f$ (Theorem 2.11., 1) and 2)). In order to prove 4), let us denote the minimal values of (P) and $(\mathrm{P})^{(qc)}$ by m resp. $m^{(qc)}$. On the one hand, from quasiconvexity of $f^{(qc)}$ it follows that

$$f^{(qc)}(\mathfrak{o}) = \inf\Big\{\frac{1}{|\,\Omega\,|}\int_{\Omega} f^{(qc)}(\,Jx(t)\,)\,dt \;\Big|\; x\in W_0^{1,\infty}(\Omega,\mathbb{R}^n)\,,\; Jx(t)\in\mathrm{K}\;\;(\forall)\,t\in\Omega\Big\} = \frac{m^{(qc)}}{|\,\Omega\,|}\,, \tag{3.10}$$

on the other hand, from Definition 2.6. we get

$$f^{(qc)}(\mathfrak{o}) = f^{*}(\mathfrak{o}) = \inf\Big\{\frac{1}{|\,\Omega\,|}\int_{\Omega} f(\,Jx(t)\,)\,dt \;\Big|\; x\in W_0^{1,\infty}(\Omega,\mathbb{R}^n)\,,\; Jx(t)\in\mathrm{K}\;\;(\forall)\,t\in\Omega\Big\} = \frac{m}{|\,\Omega\,|}\,; \tag{3.11}$$

consequently, we have $m = m^{(qc)}$. The proof of property 3) will be divided into seven steps.

• **Step 1.** *Two preliminary lemmata.*

Lemma 3.5. *Let functions $f \in \mathcal{F}_{\mathrm{K}}$ and $u \in L^{\infty}(\Omega,\mathbb{R}^{nm})$ with $u(t)\in\mathrm{K}$ $(\forall)\,t\in\Omega$ be given. Assume that the uniform continuity of f on K is described through the ε-δ relation*

$$\big|\,v'-v''\,\big| \leqslant \delta(\varepsilon) < 1 \implies \big|\,f(v')-f(v'')\,\big| \leqslant \varepsilon \quad \forall\,v',\,v''\in\mathrm{K}\,. \tag{3.12}$$

Then we have the implication

$$0 < \gamma < \delta_3(\varepsilon) = \mathrm{Min}\Big(\frac{1}{2}\,, \frac{1}{(c_{\mathrm{K}})^2}\,, \Big(\frac{\delta(\varepsilon)\cdot c_{\mathrm{K}}}{18\,(C_{\mathrm{K}})^2}\Big)^2\Big) \tag{3.13}$$

$$\implies \int_{\Omega}\Big(f^{(qc)}(u(t)) - f^{(qc)}(\,(1-\gamma)\,u(t)\,)\Big)\,dt \geqslant -4\,|\,\Omega\,|\,\varepsilon$$

where c_{K} and C_{K} are the quantities defined at the beginning of the subsection. The relation between $\delta_3(\varepsilon)$ and ε does not depend on u.

Proof. Let $0 < \gamma < \frac{1}{2}$. For all points $v \in \mathrm{K} \setminus (1-\sqrt{\gamma}\,)\,\mathrm{K}$ it holds:

$$\big|\,v-(1-\gamma)\,v\,\big| \leqslant \big|\,v-(1-\gamma)\,(1-\sqrt{\gamma}\,)\,v\,\big| \leqslant C_{\mathrm{K}}\,\big(1-(1-\gamma)\,(1-\sqrt{\gamma}\,)\big) \tag{3.14}$$

$$\leqslant C_{\mathrm{K}}\,\big(\sqrt{\gamma}+\sqrt{\gamma^2}+\sqrt{\gamma^3}\,\big) \leqslant 3\sqrt{\gamma}\,C_{\mathrm{K}}\,.$$

From Theorem 2.8., 3) we take the estimate

$$3\sqrt{\gamma}\,C_{\mathrm{K}} \leqslant \frac{\delta(\varepsilon)}{6}\cdot\frac{c_{\mathrm{K}}}{C_{\mathrm{K}}} \implies \gamma \leqslant \Big(\frac{\delta(\varepsilon)\cdot c_{\mathrm{K}}}{18\,(C_{\mathrm{K}})^2}\Big)^2 \implies f^{(qc)}(v) - f^{(qc)}(\,(1-\gamma)v\,) \geqslant -2\,\varepsilon\,. \tag{3.15}$$

Consider now points $v \in (1-\sqrt{\gamma}\,)\,\mathrm{K}$. These obey

$$\big|\,v-(1-\gamma)v\,\big| \leqslant \gamma\,C_{\mathrm{K}} \quad\text{as well as}\quad \mathrm{Dist}\,(\,(1-\gamma)v\,,\partial\mathrm{K}\,) \geqslant \mathrm{Dist}\,(v\,,\partial\mathrm{K}) \geqslant c_{\mathrm{K}}\sqrt{\gamma}\,. \tag{3.16}$$

From Theorem 2.7. it follows: If $\sqrt{\gamma}\, c_{\mathrm{K}} \leqslant 1$, i. e. $\gamma \leqslant 1/(c_{\mathrm{K}})^2$, as well as

$$\gamma\, C_{\mathrm{K}} \;\leqslant\; \frac{\delta(\varepsilon)}{4\, C_{\mathrm{K}}} \cdot \mathrm{Min}\Big(1\,,\; \sqrt{\gamma}\, c_{\mathrm{K}}\,,\; \sqrt{\gamma}\, c_{\mathrm{K}}\Big) \;\leqslant\; \frac{\delta(\varepsilon)}{4\, C_{\mathrm{K}}} \cdot \mathrm{Min}\Big(1\,, \mathrm{Dist}\,(v\,,\,\partial\mathrm{K})\,,\, \mathrm{Dist}\,(\,(1-\gamma)\, v\,,\,\partial\mathrm{K})\Big) \tag{3.17}$$

then we get the implications

$$\gamma \;\leqslant\; \frac{\delta(\varepsilon)}{4(C_{\mathrm{K}})^2} \cdot \sqrt{\gamma}\, c_{\mathrm{K}} \quad\Longrightarrow\quad \gamma \;\leqslant\; \Big(\,\frac{\delta(\varepsilon)\cdot c_{\mathrm{K}}}{4\,(C_{\mathrm{K}})^2}\,\Big)^2 \quad\Longrightarrow\quad \big|\, f^{(qc)}(v) -\, f^{(qc)}(\,(1-\gamma)\, v\,)\,\big| \;\leqslant\; 2\,\varepsilon\,. \tag{3.18}$$

After the decomposition $\Omega = \Omega_1 \cup \Omega_2$ with

$$\Omega_1 = \big\{\, t\in\Omega \;\big|\; u(t)\in \mathrm{K} \setminus (1-\sqrt{\gamma}\,)\,\mathrm{K}\,\big\}\,,\;\; \Omega_2 = \big\{\, t\in\Omega \;\big|\; u(t)\in (1-\sqrt{\gamma}\,)\,\mathrm{K}\,\big\}\,, \tag{3.19}$$

we arrive at (3.20)

$$\int_\Omega \Big(\, f^{(qc)}(\,u(t)\,) - f^{(qc)}(\,(1-\gamma)\,u(t)\,)\,\Big)\,dt \;\geqslant\; \int_{\Omega_1} \Big(\;\ldots\;\Big)\,dt - \int_{\Omega_2} \Big|\;\ldots\;\Big|\,dt \geqslant -2\,|\,\Omega_1\,|\,\varepsilon - 2\,|\,\Omega_2\,|\,\varepsilon \geqslant -4\,|\,\Omega\,|\,\varepsilon\,. \quad \blacksquare$$

Lemma 3.6. *Let functions $f \in \mathcal{F}_{\mathrm{K}}$ and $u \in L^\infty(\Omega, \mathbb{R}^{nm})$ with $u(t) \in \mathrm{K}$ $(\forall)\, t \in \Omega$ be given. Then for arbitrary $\varepsilon > 0$ there exists an index $N_1(\varepsilon) \in \mathbb{N}$ with*

$$\Big|\int_\Omega f^{(qc)}(\,(1-\frac{1}{N})\,u(t)\,)\,dt - \int_\Omega f^{(qc)}(\,u(t)\,)\,dt\Big| \;\leqslant\; |\,\Omega\,|\,\varepsilon \quad \forall\, N \geqslant N_1(\varepsilon)\,. \tag{3.21}$$

Proof. From the radial continuity of $f^{(qc)}$ (Theorems 2.13. and 2.8., 2)), we get $f^{(qc)}(\,u(t)\,) = \lim_{N\to\infty} f^{(qc)}(\,(1-1/N)\,u(t)\,)$ as a pointwise limit for all $t\in\Omega$. Then the assertion follows from Lebesgue's dominated convergence theorem. $\blacksquare$

• **Step 2.** *Decomposition of the integrals.* Consider a sequence $\{\, x^N \,\}$, $W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ of functions admissible in (P), which converge to a limit element $\hat{x} \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)$ in the sense of 3), and a number $\varepsilon > 0$. Choose now $0 < \gamma < \mathrm{Min}\,\big(\,\delta_3(\varepsilon)\,,\, 1/N_1(\varepsilon)\,\big)$ according to Lemma 3.5. and 3.6. and decompose:

$$\begin{aligned}
&\int_\Omega \Big(\, f^{(qc)}(\,Jx^N(t)\,) - f^{(qc)}(\,J\hat{x}(t)\,)\,\Big)\,dt \;=\; \int_\Omega \Big(\, f^{(qc)}(\,Jx^N(t)\,) - f^{(qc)}(\,(1-\gamma)\,Jx^N(t)\,)\,\Big)\,dt \\
&+ \int_\Omega \Big(\, f^{(qc)}(\,(1-\gamma)\,Jx^N(t)\,) - f^{(qc)}(\,(1-\gamma)\,J\hat{x}(t)\,)\,\Big)\,dt \;+\; \int_\Omega \Big(\, f^{(qc)}(\,(1-\gamma)\,J\hat{x}(t)\,) - f^{(qc)}(\,J\hat{x}(t)\,)\,\Big)\,dt
\end{aligned} \tag{3.22}$$

$$\geqslant \int_\Omega \Big(\, f^{(qc)}(\,(1-\gamma)\,Jx^N(t)\,) - f^{(qc)}(\,(1-\gamma)\,J\hat{x}(t)\,)\,\Big)\,dt \;-\; 5\,|\,\Omega\,|\,\varepsilon\,. \tag{3.23}$$

Let us define

$$z^N(t) = (1-\gamma)\,x^N(t) \quad\Longrightarrow\quad Jz^N(t) = (1-\gamma)\,Jx^N(t)\,; \tag{3.24}$$

$$\hat{z}(t) = (1-\gamma)\,\hat{x}(t) \quad\Longrightarrow\quad J\hat{z}(t) = (1-\gamma)\,J\hat{x}(t)\,; \tag{3.25}$$

$$x^N \xrightarrow{\;*\;}{}_{L^\infty(\Omega,\mathbb{R}^n)}\, \hat{x} \quad\Longrightarrow\quad z^N \xrightarrow{\;*\;}{}_{L^\infty(\Omega,\mathbb{R}^n)}\, \hat{z}\,; \tag{3.26}$$

$$Jx^N \xrightarrow{\;*\;}{}_{L^\infty(\Omega,\mathbb{R}^{nm})}\, J\hat{x} \quad\Longrightarrow\quad Jz^N \xrightarrow{\;*\;}{}_{L^\infty(\Omega,\mathbb{R}^{nm})}\, J\hat{z}\,; \tag{3.27}$$

$$Jx^N(t) \in \mathrm{K}\;\; (\forall)\, t\in\Omega\;\; \forall\, N\in\mathbb{N} \quad\Longrightarrow\quad Jz^N(t) \in (1-\gamma)\,\mathrm{K}\;\; (\forall)\, t\in\Omega\;\; \forall\, N\in\mathbb{N}\,; \tag{3.28}$$

$$J\hat{x}(t) \in \mathrm{K}\;\; (\forall)\, t\in\Omega \quad\Longrightarrow\quad J\hat{z}(t) \in (1-\gamma)\,\mathrm{K}\;\; (\forall)\, t\in\Omega\,. \tag{3.29}$$

According to Lemma 3.4., to the index $N \in \mathbb{N}$ and the given $\varepsilon > 0$ we choose finitely many closed cubes $\mathrm{Q}_s \subseteq \Omega$ with centers t_s and edge length $0 \leqslant \delta_s \leqslant 1/N$ which are mutually disjoint and satisfy

$$\big|\, \Omega \setminus \textstyle\bigcup_{s=1}^{r} \mathrm{Q}_s \,\big| \leqslant \varepsilon\,; \tag{3.30}$$

$$\big|\, J\hat{z}(t) - M_s \,\big| \leqslant \delta_4(\varepsilon) \,=\, \mathrm{Min}\Big(\,\varepsilon\,,\, \frac{\delta(\varepsilon)}{4\,C_{\mathrm{K}}} \cdot \mathrm{Min}\big(\,1\,,\, \frac{\gamma\,c_{\mathrm{K}}}{2}\,\big)\Big) \quad (\forall)\, t \in \mathrm{Q}_s\,,\ 1 \leqslant s \leqslant r \tag{3.31}$$

where $\delta(\varepsilon)$ has been taken from ε-δ relation of the uniform continuity of f on K and M_s denote the (n, m)-matrices of the mean values

$$M_s \,=\, \begin{pmatrix} \dfrac{1}{|\,\mathrm{Q}_s\,|} \displaystyle\int_{\mathrm{Q}_s} \frac{\partial \hat{z}_1}{\partial t_1}(\tau)\, d\tau & \dots & \dfrac{1}{|\,\mathrm{Q}_s\,|} \displaystyle\int_{\mathrm{Q}_s} \frac{\partial \hat{z}_1}{\partial t_m}(\tau)\, d\tau \\ \vdots & & \vdots \\ \dfrac{1}{|\,\mathrm{Q}_s\,|} \displaystyle\int_{\mathrm{Q}_s} \frac{\partial \hat{z}_n}{\partial t_1}(\tau)\, d\tau & \dots & \dfrac{1}{|\,\mathrm{Q}_s\,|} \displaystyle\int_{\mathrm{Q}_s} \frac{\partial \hat{z}_n}{\partial t_m}(\tau)\, d\tau \end{pmatrix}. \tag{3.32}$$

Using these notations, we decompose further:

$$\int_{\Omega} \Big(\, f^{(qc)}(\, Jx^N(t)\,) - f^{(qc)}(\, J\hat{x}(t)\,)\Big)\, dt \,\geqslant\, -5\,|\,\Omega\,|\,\varepsilon \,+\, \int_{\Omega} \Big(\, f^{(qc)}(\, Jz^N(t)\,) - f^{(qc)}(\, J\hat{z}(t)\,)\Big)\, dt \tag{3.33}$$

$$=\, -5\,|\,\Omega\,|\,\varepsilon \,+\, J_1(N) \,+\, J_2(N) \,+\, J_3(N) \,+\, J_4(N) \text{ with}$$

$$J_1(N) \,=\, \int_{\Omega \setminus \bigcup_s \mathrm{Q}_s} \Big(\, f^{(qc)}(\, Jz^N(t)\,) - f^{(qc)}(\, J\hat{z}(t)\,)\Big)\, dt\,; \tag{3.34}$$

$$J_2(N) \,=\, \sum_s \int_{\mathrm{Q}_s} \Big(\, f^{(qc)}(\, J\hat{z}(t) + (Jz^N(t) - J\hat{z}(t))\,) - f^{(qc)}(\, M_s - (Jz^N(t) - J\hat{z}(t))\,)\Big)\, dt\,; \tag{3.35}$$

$$J_3(N) \,=\, \sum_s \int_{\mathrm{Q}_s} \Big(\, f^{(qc)}(\, M_s - (Jz^N(t) - J\hat{z}(t))\,) - f^{(qc)}(\, M_s\,)\Big)\, dt\,; \tag{3.36}$$

$$J_4(N) \,=\, \sum_s \int_{\mathrm{Q}_s} \Big(\, f^{(qc)}(\, M_s\,) - f^{(qc)}(\, J\hat{z}(t)\,)\Big)\, dt\,. \tag{3.37}$$

• **Step 3.** *Investigation of* $J_1(N)$. Together with $f \in \mathcal{F}_{\mathrm{K}}$, f^c is bounded from below on K,[34] and since $f^c \leqslant f^{(qc)} \leqslant f$ there exists a constant $C_1 > 0$ with $|\, f^{(qc)}(v)\,| \leqslant C_1\ \forall\, v \in \mathrm{K}$. It follows that

$$J_1(N) \,\geqslant\, -\int_{\Omega \setminus \bigcup_s \mathrm{Q}_s} \Big|\, f^{(qc)}(\, Jz^N(t)\,) - f^{(qc)}(\, J\hat{z}(t)\,)\Big|\, dt \,\geqslant\, -2\,C_1 \cdot \big|\, \Omega \setminus \textstyle\bigcup_{s=1}^{r} \mathrm{Q}_s \,\big| \,\geqslant\, -2\,C_1\,\varepsilon\,. \tag{3.38}$$

• **Step 4.** *Investigation of* $J_2(N)$. For all $N \in \mathbb{N}$, it holds that $\mathrm{Dist}\,(Jz^N(t)\,,\, \partial\mathrm{K}) \geqslant \gamma\, c_{\mathrm{K}}$, and from the choice of the cubes Q_s it follows that

$$\big|\,\big(\, J\hat{z}(t) + (Jz^N(t) - J\hat{z}(t))\big) - \big(\, M_s - (Jz^N(t) - J\hat{z}(t))\big)\,\big| \,=\, \big|\, J\hat{z}(t) - M_s \,\big| \,\leqslant\, \delta_4(\varepsilon) \,\leqslant\, \frac{\gamma\,c_{\mathrm{K}}}{2}\,; \tag{3.39}$$

consequently, we find $\mathrm{Dist}\,(M_s - (Jz^N(t) - J\hat{z}(t))\,,\, \partial\mathrm{K}) \geqslant \frac{1}{2}\,\gamma\, c_{\mathrm{K}}$ (for almost all $t \in \mathrm{Q}_s$, respectively). Furthermore, from Theorem 2.7. it follows for almost all $t \in \mathrm{Q}_s$:

$$\big|\,\big(\, J\hat{z}(t) + (Jz^N(t) - J\hat{z}(t))\big) - \big(\, M_s - (Jz^N(t) - J\hat{z}(t))\big)\,\big| \,\leqslant\, \delta_4(\varepsilon) \,\leqslant\, \frac{\delta(\varepsilon)}{4\,C_{\mathrm{K}}} \cdot \mathrm{Min}\Big(\,1\,,\, \frac{\gamma\,c_{\mathrm{K}}}{2}\,,\, \gamma\,c_{\mathrm{K}}\Big) \tag{3.40}$$

$$\leqslant\, \frac{\delta(\varepsilon)}{4\,C_{\mathrm{K}}} \cdot \mathrm{Min}\,\big(\,1\,,\, \mathrm{Dist}\,(M_s - (Jz^N(t) - J\hat{z}(t))\,,\, \partial\mathrm{K})\,,\, \mathrm{Dist}\,(Jz^N(t),\, \partial\mathrm{K})\big) \tag{3.41}$$

$$\Longrightarrow \quad \big|\, f^{(qc)}(\, J\hat{z}(t) - (Jz^N(t) - J\hat{z}(t))\,) - f^{(qc)}(\, M_s - (Jz^N(t) - J\hat{z}(t))\,)\,\big| \,\leqslant\, 2\,\varepsilon\,.$$

[34] [Dacorogna 89] , p. 42, Corollary 2.9.

Summing up, we arrive at

$$J_2(N) \geqslant -\sum_s \int_{\mathrm{Q}_s} \Big| f^{(qc)}(\, J\hat{z}(t) + (Jz^N(t) - J\hat{z}(t))\,) - f^{(qc)}(\, M_s - (Jz^N(t) - J\hat{z}(t))\,) \Big|\, dt \tag{3.42}$$
$$\geqslant -\sum_s |\,\mathrm{Q}_s\,| \cdot 2\,\varepsilon \geqslant -2\,|\,\Omega\,|\,\varepsilon\,.$$

• **Step 5.** *Investigation of* $J_3(N)$. In this step, we change again the notations and define

$$w^N(t) = z^N(t) - \hat{z}(t)\,, \quad Jw^N(t) = Jz^N(t) - J\hat{z}(t) \quad \text{with } w^N \in W_0^{1,\infty}(\Omega, \mathbb{R}^n)\,; \tag{3.43}$$
$$z^N \xrightarrow{\;*\;} {}_{L^\infty(\Omega,\mathbb{R}^n)}\, \hat{z} \quad \Longrightarrow \quad w^N \xrightarrow{\;*\;} {}_{L^\infty(\Omega,\mathbb{R}^n)}\, \mathfrak{o}\,; \tag{3.44}$$
$$Jz^N \xrightarrow{\;*\;} {}_{L^\infty(\Omega,\mathbb{R}^{nm})}\, J\hat{z} \quad \Longrightarrow \quad Jw^N \xrightarrow{\;*\;} {}_{L^\infty(\Omega,\mathbb{R}^{nm})}\, \mathfrak{o}\,. \tag{3.45}$$

From $w^N \xrightarrow{\;*\;} {}_{L^\infty(\Omega,\mathbb{R}^n)}\, \mathfrak{o}$ and $Jw^N \xrightarrow{\;*\;} {}_{L^\infty(\Omega,\mathbb{R}^{nm})}\, \mathfrak{o}$, it follows $\|\, w^N \,\|_{L^\infty(\Omega,\mathbb{R}^n)} \to 0$.[35] In order to exploit the quasiconvexity of $f^{(qc)}$, we must change the values of w^N on the boundaries $\partial\mathrm{Q}_s$ into zero. This can be done as follows. Let us choose closed cubes $\mathrm{Q}_s^0 \subset \mathrm{int}\,(\mathrm{Q}_s)$ with the same center as Q_s and $|\,\mathrm{Q}_s \setminus \mathrm{Q}_s^0\,| \leqslant \varepsilon \cdot |\,\mathrm{Q}_s\,|$, respectively. Denote then $\mathrm{Dist}\,(\partial\mathrm{Q}_s^0\,,\,\partial\mathrm{Q}_s) = \varrho_s$. Further, we define functions $\varphi_s \in C^\infty(\mathrm{Q}_s, \mathbb{R})$ with

$$\varphi_s(t) \begin{cases} = 1 & \big|\; t \in \mathrm{Q}_s^0; \\ \in [\,0\,,\,1\,] & \big|\; t \in \mathrm{Q}_s \setminus \mathrm{Q}_s^0; \\ = 0 & \big|\; t \in \partial\mathrm{Q}_s \end{cases} \tag{3.46}$$

as well as $|\,\nabla\varphi_s(t)\,| \leqslant C_2/\varrho_s$ where $C_2 > 0$ is a constant. Now we get from quasiconvexity of $f^{(qc)}$:

$$\int_{\mathrm{Q}_s} f^{(qc)}(\, M_s\,)\, dt \leqslant \int_{\mathrm{Q}_s} f^{(qc)}(\, M_s + J(\, \varphi_s^k(t) \cdot w^N(t)\,)\,)\, dt \tag{3.47}$$
$$= \int_{\mathrm{Q}_s^0} f^{(qc)}(\, M_s + Jw^N(t)\,)\, dt + \int_{\mathrm{Q}_s \setminus \mathrm{Q}_s^0} f^{(qc)}(\, M_s + \varphi_s(t)\, Jw^N(t) + w^N(t)\, \nabla\varphi_s(t)^{\mathrm{T}}\,)\, dt\,.$$

By Step 3, it holds $M_s \in (1 - \frac{1}{2}\,\gamma)\,\mathrm{K}$ as well as $M_s + Jw^N(t) \in (1 - \frac{1}{2}\,\gamma)\,\mathrm{K}$ for all s and for almost all $t \in \mathrm{Q}_s$. Since $0 \leqslant \varphi_s(t) \leqslant 1$, we find

$$M_s + \varphi_s(t)\, Jw^N(t) \in \big[\, M_s\,,\, M_s + Jw^N(t)\,\big] \subset (1 - \frac{\gamma}{2})\,\mathrm{K}\,. \tag{3.48}$$

Moreover, we get with a further constant $C_3 > 0$

$$\big|\, w^N(t)\, \nabla\varphi_s(t)^{\mathrm{T}}\,\big| \leqslant C_3 \cdot \big|\, \nabla\varphi_s(t)\,\big| \cdot \|\, w^N \,\|_{L^\infty(\Omega,\mathbb{R}^n)} \leqslant \frac{C_2\, C_3}{\varrho_s} \cdot \|\, w^N \,\|_{L^\infty(\Omega,\mathbb{R}^n)}\,. \tag{3.49}$$

Consequently, for every s there exists $N_{2,s}(\varepsilon) \in \mathbb{N}$ with $|\, w^N(t)\, \nabla\varphi_s(t)^{\mathrm{T}}\,| \leqslant \frac{1}{4}\,\gamma\, c_{\mathrm{K}}$ for all $N \geqslant N_{2,s}(\varepsilon)$, and for $N \geqslant \mathrm{Max}_{\,s}\; N_{2,s}(\varepsilon)$, in both integrands on the right-hand side of (3.47) the arguments belong to K. Thus, inequality (3.47) may be transformed for the respective indices N as follows:

$$\int_{\mathrm{Q}_s} \Big(f^{(qc)}(\, M_s + Jw^N(t)\,) - f^{(qc)}(\, M_s\,) \Big)\, dt \tag{3.50}$$
$$\geqslant -\int_{\mathrm{Q}_s \setminus \mathrm{Q}_s^0} f^{(qc)}(\, M_s + Jw^N(t)\,)\, dt - \int_{\mathrm{Q}_s \setminus \mathrm{Q}_s^0} f^{(qc)}(\, M_s + \varphi_s(t)\, Jw^N(t) + w^N(t)\, \nabla\varphi_s(t)^{\mathrm{T}}\,)\, dt$$
$$\geqslant -\int_{\mathrm{Q}_s \setminus \mathrm{Q}_s^0} \big|\, \ldots\,\big|\, dt - \int_{\mathrm{Q}_s \setminus \mathrm{Q}_s^0} \big|\, \ldots\,\big|\, dt \geqslant -2\, C_1\, \big|\, \mathrm{Q}_s\,\big|\, \varepsilon\,. \tag{3.51}$$

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[35] Cf. [Dacorogna 04], p. 36, Corollary 1.45.

Summing up, we arrive at

$$J_3(N) \;=\; \sum_s \int_{\mathrm{Q}_s} \Big(f^{(qc)}(\, M_s + Jw^N(t)\,) - f^{(qc)}(\, M_s\,) \Big)\, dt \;\geqslant\; -2\, C_1 \,\big|\, \Omega \,\big|\, \varepsilon \quad \forall\, N \geqslant \operatorname*{Max}_s\, N_{2,s}(\varepsilon)\,. \tag{3.52}$$

• **Step 6.** *Investigation of* $J_4(N)$. Analogously to Step 4, from $\mathrm{Dist}\,(J\hat{z}(t)\,,\,\partial\mathrm{K}) \geqslant \gamma\, c_{\mathrm{K}}$ and $|\, J\hat{z}(t) - M_s\,| \leqslant \delta_4(\varepsilon) \leqslant \frac{1}{2}\,\gamma\, c_{\mathrm{K}}$ we get $\mathrm{Dist}\,(M_s\,,\,\partial\mathrm{K}) \geqslant \frac{1}{2}\,\gamma\, c_{\mathrm{K}}$ for almost all $t \in \mathrm{Q}_s$. From Theorem 2.7., it follows again that

$$\begin{aligned} \big|\, M_s - J\hat{z}(t)\,\big| \;\leqslant\; \delta_4(\varepsilon) \;\leqslant\; \frac{\delta(\varepsilon)}{4\, C_{\mathrm{K}}} \cdot \mathrm{Min}\Big(\, 1\,,\, \frac{\gamma\, c_{\mathrm{K}}}{2}\,,\, \gamma\, c_{\mathrm{K}} \Big) \qquad\qquad\qquad\qquad & \qquad (3.53) \\ \leqslant\; \frac{\delta(\varepsilon)}{4\, C_{\mathrm{K}}} \cdot \mathrm{Min}\big(\, 1\,,\, \mathrm{Dist}\,(M_s,\, \partial\mathrm{K})\,,\, \mathrm{Dist}\,(J\hat{z}(t),\, \partial\mathrm{K})\,\big) \quad \Longrightarrow \quad & \big|\, f^{(qc)}(\, M_s\,) - f^{(qc)}(\, J\hat{z}(t)\,)\,\big| \;\leqslant\; 2\,\varepsilon \end{aligned}$$

for almost all $t \in \mathrm{Q}_s$. Thus we arrive at

$$J_4(N) \;\geqslant -\sum_s \int_{\mathrm{Q}_s} \Big|\, f^{(qc)}(\, M_s\,) - f^{(qc)}(\, J\hat{z}(t)\,)\,\Big|\, dt \;\geqslant\; -\sum_s \,|\, \mathrm{Q}_s\,| \cdot 2\,\varepsilon \;\geqslant -2\,|\,\Omega\,|\,\varepsilon\,. \tag{3.54}$$

• **Step 7.** *Conclusion.* For $\varepsilon > 0$ and $N \geqslant \mathrm{Max}\big(\, N_1(\varepsilon)\,,\, N_{2,1}(\varepsilon)\,,\, \dots\,,\, N_{2,r}(\varepsilon)\big)$, the inequality

$$\begin{aligned} \int_\Omega \Big(f^{(qc)}(\, Jx^N(t)\,) - f^{(qc)}(\, J\hat{x}(t)\,)\Big)\, dt \;\geqslant\; -5\,\big|\,\Omega\,\big|\,\varepsilon \,+\, J_1(N) \,+\, J_2(N) \,+\, J_3(N) \,+\, J_4(N) \qquad & (3.55) \\ \geqslant\; -5\,\big|\,\Omega\,\big|\,\varepsilon \,-\, 2\,C_1\,\varepsilon \,-\, 2\,\big|\,\Omega\,\big|\,\varepsilon \,-\, 2\,C_1\,\big|\,\Omega\,\big|\,\varepsilon \,-\, 2\,\big|\,\Omega\,\big|\,\varepsilon \;=\; -C_4\,\varepsilon\,, & \end{aligned}$$

holds wherein the constant $C_4 > 0$ depends neither on N nor on ε. Consequently, we get for arbitrary $\varepsilon > 0$ the estimate

$$\liminf_{N\to\infty} \int_\Omega \Big(f^{(qc)}(\, Jx^N(t)\,) - f^{(qc)}(\, J\hat{x}(t)\,)\Big)\, dt \;\geqslant\; -C_4\,\varepsilon \tag{3.56}$$

from which the claimed relation $\liminf_{N\to\infty} F^{(qc)}(x^N) \geqslant F^{(qc)}(\hat{x})$ follows. The proof of Theorem 1.3. is complete. ■

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